Existence for a calibrated regime-switching local volatility model

Alexandre Zhou
Based on joint works with Benjamin Jourdain

CERMICS, Ecole des Ponts ParisTech

Les Probabilités de Demain
Institut Henri Poincaré, 3rd May, 2018

Plan

Processes matching marginal distributions

- Assume that the market gives us the prices of European call options $C(T, K)$ for all $T, K \geq 0$, on the underlying asset S
- For hedging purposes, we want a model $(S_t)_{t \geq 0}$ calibrated to those prices:

$$\forall T, K \geq 0, \;\; C(T, K) = \mathbb{E}\left[e^{-rT}\left(S_T - K\right)^+\right]$$

- By Breeden and Litzenberger (1978), marginal laws are equivalent to market prices of European Calls $C(T, K)$
- Stochastic processes matching given marginals is a question arising in mathematical finance
- Dupire calibrated Local Volatility model (1992):

$$dS_t = rS_t dt + \sigma_{Dup}(t, S_t) S_t dW_t$$

LSV models

- Dupire's model gives a perfect fit to the market prices of call options, but forward laws are unrealistic
- Motivation: get processes with richer dynamics and satisfying the same marginal constraints
- Lipton (2002) and Piterbarg (2006): Local and Stochastic Volatility (LSV) model

$$dS_t = rS_t dt + f(Y_t)\sigma(t, S_t)S_t dW_t$$

- 'Adding uncertainty' to LV models by a random multiplicative factor $f(Y_t) > 0$, where $(Y_t)_{t\geq 0}$ is a stochastic process

Gyongy's Theorem

Let X be an Ito process satisfying

$$dX_t = \alpha(t,\omega)dt + \beta(t,\omega)dW_t$$

where α, β are adapted processes. Under mild assumptions, there exists a Markov process Y_t satisfying

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dW_t$$

where X_t, Y_t have the same distribution for all $t \geq 0$ and Y can be constructed with

$$\begin{aligned} a(t,y) =& \mathbb{E}[\alpha(t,\omega)|X_t = y] \\ b^2(t,y) =& \mathbb{E}[\beta^2(t,\omega)|X_t = y] \end{aligned}$$

Calibration of LSV Models

- The LSV model is calibrated to $(C(T,K))_{T,K\geq 0}$ if

$$\mathbb{E}\left[(f(Y_t)\sigma(t,S_t)S_t)^2|S_t = x\right] = (\sigma_{Dup}(t,x)x)^2$$

$$\sigma(t,x) = \frac{\sigma_{Dup}(t,x)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t = x\right]}}$$

- The obtained SDE is nonlinear in the sense of McKean:

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}}\sigma_{Dup}(t,S_t)S_t dW_t.$$

- Open problems : Global existence and uniqueness to LSV models ? Convergence of the particles method used to simulate the SDE nonlinear in the sense of McKean ?

A RSLV model

- We consider the following dynamics (RSLV):

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t\right]}}\sigma_{Dup}(t, S_t)S_t dW_t$$

where $(Y_t)_{t\geq 0}$ takes values in $\mathcal{Y} = \{y_1, ..., y_d\}$, and

$$\mathbb{P}\left(Y_{t+dt} = y_j | Y_t = y_i, \log S_t = x\right) = q_{ij}(x)dt$$

- Switching diffusion, special case of LSV model
- Jump distributions and intensities are functions of the asset level

Existence to SDE (RSLV)

Theorem

Condition (C): there exists a symmetric positive definite $\Gamma \in \mathbb{R}^{d\times d}$ such that for all $k \in \{1,\dots,d\}$, the $d \times d$ matrix

$$\Gamma^{(k)}_{ij} = \frac{f^2(y_i)+f^2(y_j)}{2}(\Gamma_{ij}+\Gamma_{kk}-\Gamma_{ik}-\Gamma_{jk}) \text{ is positive definite on } e_k^{\perp}.$$

Under (C) and regularity conditions on σ_{Dup}, q, there exists a weak solution to the SDE (RSLV).

If $d=2$, (C) is satisfied : choice $\Gamma = I_2$,

if $d=3$, $(C) \Leftrightarrow \frac{1}{\beta_1\beta_2}+\frac{1}{\beta_2\beta_3}+\frac{1}{\beta_3\beta_1} > \frac{1}{4}$ with

$$\beta_1 = \left|\sqrt{\frac{f^2(y_2)}{f^2(y_3)}}-\sqrt{\frac{f^2(y_3)}{f^2(y_2)}}\right|, \beta_2 = \left|\sqrt{\frac{f^2(y_3)}{f^2(y_1)}}-\sqrt{\frac{f^2(y_1)}{f^2(y_3)}}\right|, \beta_3 = \left|\sqrt{\frac{f^2(y_1)}{f^2(y_2)}}-\sqrt{\frac{f^2(y_2)}{f^2(y_1)}}\right|$$

if $d \geq 4$, $\max_{1\leq k\leq d}\sum_{i\neq k} f^2(y_i)\sum_{i\neq k}\frac{1}{f^2(y_i)} \leq (d+1)^2 \Rightarrow (C)$: $\Gamma = I_d$.

Time discretization

Existence for calibrated LSV models seems challenging in the general case...
The time-discretized version is much easier ! $X = \log(S), \tau_t = [\frac{nt}{T}]\frac{T}{n}$

$$dX_t^n = \left(r - \frac{1}{2}\frac{f^2(Y_{\tau_t})}{\mathbb{E}\left[f^2(Y_{\tau_t})|X_{\tau_t}^n\right]}\sigma_{Dup}(\tau_t, X_{\tau_t}^n)\right)dt + \frac{f(Y_{\tau_t})}{\sqrt{\mathbb{E}\left[f^2(Y_{\tau_t})|X_{\tau_t}^n\right]}}\sigma_{Dup}(\tau_t, X_{\tau_t}^n)dW_t$$

Theorem

Under regularity conditions on f, σ_{Dup}, φ, there exists a constant $C > 0$ such that

$$\forall n \geq 0, |\mathbb{E}[\varphi(X_T^n) - \varphi(\log(S_T))]| \leq \frac{C}{n}.$$

Simulation of the SDE

The idea (Guyon, Henry Labordère 2008): kernel approximation (for instance, Gaussian) of the conditional expectation and interacting particles method. For $1 \leq i \leq N$,

$$dX_t^{n,i,N} = \left(r - \frac{1}{2} \frac{f^2(Y_{\tau_t})}{E_i\left[f^2(Y_{\tau_t})|X_{\tau_t}^{n,i,N}\right]} \sigma_{Dup}(\tau_t, X_{\tau_t}^{n,i,N}) \right) dt + \frac{f(Y_{\tau_t})}{\sqrt{E_i\left[f^2(Y_{\tau_t})|X_{\tau_t}^{n,i,N}\right]}} \sigma_{Dup}(\tau_t, X_{\tau_t}^{n,i,N}) dW_t^i,$$

with for $\delta > 0$ small,

$$E_i\left[f^2(Y_{\tau_t})|X_{\tau_t}^{n,i,N}\right] = \frac{\frac{1}{N}\sum_{i=1}^N f^2(Y_{\tau_t}^{n,i,N}) G_\delta(X_{\tau_t}^{n,j,N} - X_{\tau_t}^{n,i,N})}{\frac{1}{N}\sum_{i=1}^N G_\delta(X_{\tau_t}^{n,j,N} - X_{\tau_t}^{n,i,N})}$$

Speed of convergence ?

Thank you!

Thank you for your attention!