Mean field limits for Hawkes processes in a diffusive regime

Xavier Erny [1]

with Eva Löcherbach [2] and Dasha Loukianova [1]

[1]Université d'Evry Val d'Essonne (LaMME)
[2]Université Paris 1 Panthéon-Sorbonne (SAMM)

Les Probabilités de Demain, 14 juin 2019

Hawkes processes

Point process = Jump process

Hawkes processes

$$\begin{aligned} \text{Point process} &= \text{Jump process} \\ &= \text{(random) Set of the jump times} \end{aligned}$$

Hawkes processes

$$\begin{aligned} \text{Point process} &= \text{Jump process} \\ &= \text{(random) Set of the jump times} \\ &= \text{(random) Point measure on } \mathbb{R}_+ \end{aligned}$$

Hawkes processes

$$\begin{aligned} \text{Point process} &= \text{Jump process} \\ &= \text{(random) Set of the jump times} \\ &= \text{(random) Point measure on } \mathbb{R}_+ \end{aligned}$$

Hawkes processes = Interacting point processes on $\mathbb{R}_+$

Hawkes processes

Point process = Jump process
= (random) Set of the jump times
= (random) Point measure on $\mathbb{R}_+$

Hawkes processes = Interacting point processes on $\mathbb{R}_+$

Example : 2 processes Z_1 and Z_2

Z_1 inhibits Z_2
Z_2 self-excitation

Modeling in neurosciences

Neural activity $=$ Set of spike times

Modeling in neurosciences

Neural activity = Set of spike times
= Point process

Modeling in neurosciences

Neural activity = Set of spike times
= Point process

Spike rate depends on the potential of the neuron

Modeling in neurosciences

$$\begin{aligned}\text{Neural activity} &= \text{Set of spike times}\\ &= \text{Point process}\end{aligned}$$

Spike rate depends on the potential of the neuron

Each spike modifies the potential of the neurons

Contents

Stochastic Intensity

Z point process on $\mathbb{R}_+$

$\lambda \; : \mathbb{R}_+ \to \mathbb{R}_+$ stochastic process

Stochastic Intensity

Z point process on $\mathbb{R}_+$

$\lambda\ : \mathbb{R}_+ \to \mathbb{R}_+$ stochastic process

λ stochastic intensity of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b])|\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda(t)dt \middle| \mathcal{F}_a\right]$$

Definition : Hawkes processes

$(Z^1, \ldots, Z^N)$ system of Hawkes processes :

- λ^i stochastic intensity of Z^i

Definition : Hawkes processes

$(Z^1, \dots, Z^N)$ system of Hawkes processes :

- λ^i stochastic intensity of Z^i
- $\lambda^i(t) = f_i\left(\sum_{j=1}^{N} \int_{[0,t[} h_{ji}(t-s)dZ^j(s)\right)$

Definition : Hawkes processes

$(Z^1, \ldots, Z^N)$ system of Hawkes processes :

- λ^i stochastic intensity of Z^i
- $$\lambda^i(t) = f_i\left(\underbrace{\sum_{j=1}^{N} \int_{[0,t[} h_{ji}(t-s)dZ^j(s)}_{X_t^{N,i}}\right)$$

$Z^i([0,t])$ = number of spikes of neuron i in $[0,t]$

$X_t^{N,i}$ = potential of neuron i at time t

f_i = spike rate function

h_{ji} = leakage function

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- $\lambda^{N,i}$ stochastic intensity of $Z^{N,i}$
- $\lambda^{N,i}(t) = f_i\left(\sum_{j=1}^{N} \int_{[0,t[} h_{ji}(t-s) \quad dZ^{N,j}(s)\right)$

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- $\lambda^{N,i}$ stochastic intensity of $Z^{N,i}$
- $\lambda^{N,i}(t) = f_i\left(\sum_{j=1}^{N} \int_{[0,t[} h_{ji}(t-s) \quad dZ^{N,j}(s)\right)$

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- λ^N stochastic intensity of $Z^{N,i}$
- $\lambda^N(t) = f\left(\sum_{j=1}^{N} \int_{[0,t[} h(t-s) dZ^{N,j}(s)\right)$

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- λ^N stochastic intensity of $Z^{N,i}$
- $\lambda^N \ (t) = f\left(\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t[} h\ (t-s) \qquad dZ^{N,j}(s)\right)$

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- λ^N stochastic intensity of $Z^{N,i}$
- $\lambda^N(t) = f\left(\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t[} h(t-s) U_j(s) dZ^{N,j}(s)\right)$

$U_j(s)$ iid with mean 0 and variance 1

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- λ^N stochastic intensity of $Z^{N,i}$
- $\lambda^N(t) = f\left(\underbrace{\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t[} h(t-s) U_j(s) dZ^{N,j}(s)}_{X_t^N}\right)$

$U_j(s)$ iid with mean 0 and variance 1

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- λ^N stochastic intensity of $Z^{N,i}$
- $\lambda^N(t) = f\left(\underbrace{\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t[} h(t-s) U_j(s) dZ^{N,j}(s)}_{X_t^N}\right)$

$U_j(s)$ iid with mean 0 and variance 1

Hawkes processes in diffusive mean field

For each $N \in \mathbb{N}^*$, we consider $\left(Z^{N,1}, \ldots, Z^{N,N}\right)$:

- λ^N stochastic intensity of $Z^{N,i}$
- $\lambda^N(t) = f\left(\underbrace{\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t[} e^{-\alpha(t-s)} U_j(s) dZ^{N,j}(s)}_{X_t^N}\right)$

$U_j(s)$ iid with mean 0 and variance 1

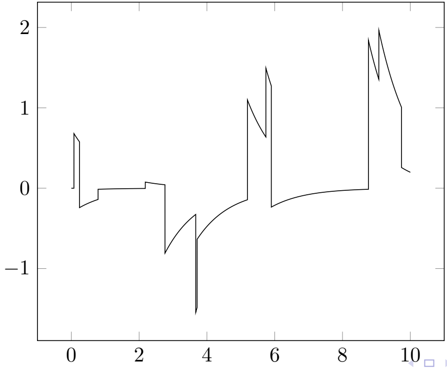
Dynamique de X^N

$$X^N_t := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]} e^{-\alpha(t-s)} U_j(s) dZ^{N,j}(s)$$

Dynamique de X^N

$$X_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]} e^{-\alpha(t-s)} U_j(s) dZ^{N,j}(s)$$

$$\begin{cases} X_t^N = X_s^N e^{-\alpha(t-s)} & \text{if none of the } Z^{N,j} \text{ charge } [s,t] \\ X_t^N = X_{t-}^N + \frac{U_j(t)}{\sqrt{N}} & \text{if } Z^{N,j} \text{ charges } t \end{cases}$$



Markov Process

$(X_t)_{t \geq 0}$ Markov process

Markov Process

$(X_t)_{t\geq 0}$ Markov process

$(P_t)_{t\geq 0}$ semigroup of X :

$$P_t g(x) \;:=\; \mathbb{E}_x\left[g(X_t)\right] \;:=\; \mathbb{E}\left[g(X_t)|X_0 = x\right]$$

Markov Process

$(X_t)_{t\geq 0}$ Markov process

$(P_t)_{t\geq 0}$ semigroup of X :

$$P_t g(x) \; := \; \mathbb{E}_x\left[g(X_t)\right] \; := \; \mathbb{E}\left[g(X_t)|X_0 = x\right]$$

A generator of X :

$$Ag(x) \; := \; \frac{d}{dt}\left(P_t g(x)\right)|_{t=0}$$

Convergence of the generators

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j(t) dZ^{N,j}(t)$$

Convergence of the generators

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j(t) dZ^{N,j}(t)$$

$$A^N g(x) = -\alpha x g'(x) + N f(x) \mathbb{E}\left[g\left(x + \frac{U}{\sqrt{N}} \right) - g(x) \right]$$

Convergence of the generators

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j(t) dZ^{N,j}(t)$$

$$A^N g(x) = -\alpha x g'(x) + N f(x) \mathbb{E}\underbrace{\left[g\left(x + \tfrac{U}{\sqrt{N}}\right) - g(x)\right]}_{\frac{U}{\sqrt{N}} g'(x) + \frac{U^2}{2N} g''(x) + O(1/N\sqrt{N})}$$

Convergence of the generators

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j(t) dZ^{N,j}(t)$$

$$A^N g(x) = -\alpha x g'(x) + N f(x) \mathbb{E}\underbrace{\left[g\left(x + \frac{U}{\sqrt{N}}\right) - g(x) \right]}_{\cancel{\frac{U}{\sqrt{N}} g'(x)} + \frac{U^2}{2N} g''(x) + O(1/N\sqrt{N})}$$

Convergence of the generators

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j(t) dZ^{N,j}(t)$$

$$A^N g(x) = -\alpha x g'(x) + N f(x) \mathbb{E}\underbrace{\left[g\left(x + \tfrac{U}{\sqrt{N}} \right) - g(x) \right]}_{\cancel{\frac{U}{\sqrt{N}} g'(x)} + \frac{U^2}{2N} g''(x) + O(1/N\sqrt{N})}$$

$$N \longrightarrow +\infty \;:\; \bar{A} g(x) = -\alpha x g'(x) + \tfrac{1}{2} f(x) g''(x)$$

Convergence of the generators

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} U_j(t) dZ^{N,j}(t)$$

$$A^N g(x) = -\alpha x g'(x) + N f(x) \mathbb{E}\underbrace{\left[g\left(x + \frac{U}{\sqrt{N}} \right) - g(x) \right]}_{\cancel{\frac{U}{\sqrt{N}} g'(x)} + \frac{U^2}{2N} g''(x) + O(1/N\sqrt{N})}$$

$$N \longrightarrow +\infty \ : \quad \bar{A} g(x) = -\alpha x g'(x) + \tfrac{1}{2} f(x) g''(x)$$

$$d\bar{X}_t = -\alpha \bar{X}_t dt + \sqrt{f(\bar{X}_t)} dB_t$$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

Sketch of proof : $u(s) = P_{t-s}^N \bar{P}_s g(x)$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

Sketch of proof : $u(s) = P_{t-s}^N \bar{P}_s g(x)$

$$\left(\bar{P}_t - P_t^N\right) g(x) = u(t) - u(0)$$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

Sketch of proof : $u(s) = P_{t-s}^N \bar{P}_s g(x)$

$$\begin{aligned}\left(\bar{P}_t - P_t^N\right) g(x) =& u(t) - u(0) \\ =& \int_0^t u'(s) ds\end{aligned}$$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

Sketch of proof : $u(s) = P_{t-s}^N \bar{P}_s g(x)$

$$\begin{aligned}
\left(\bar{P}_t - P_t^N\right) g(x) =& u(t) - u(0) \\
=& \int_0^t u'(s) ds \\
=& \int_0^t \left[-\frac{d}{du} \left(P_u^N \bar{P}_s g(x) \right) \Big|_{u=t-s} + \frac{d}{du} \left(P_{t-s}^N \bar{P}_u g(x) \right) \Big|_{u=s} \right] ds
\end{aligned}$$

Convergence of the semigroups (1)

$$\left(\bar{P}_t - P^N_t\right) g(x) = \int_0^t P^N_{t-s}\left(\bar{A} - A^N\right)\bar{P}_s g(x) ds$$

Sketch of proof : $u(s) = P^N_{t-s}\bar{P}_s g(x)$

$$\begin{aligned}\left(\bar{P}_t - P^N_t\right) g(x) =& u(t) - u(0)\\ =& \int_0^t u'(s) ds\\ =& \int_0^t \left[-\frac{d}{du}\left(P^N_u \bar{P}_s g(x)\right)\Big|_{u=t-s} + \frac{d}{du}\left(P^N_{t-s}\bar{P}_u g(x)\right)\Big|_{u=s}\right] ds\\ =& \int_0^t \left[-P^N_{t-s}A^N\bar{P}_s g(x) + P^N_{t-s}\bar{A}\bar{P}_s g(x)\right] ds\end{aligned}$$

Convergence of the semigroups (2)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

Convergence of the semigroups (2)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

$$\left|\left(\bar{P}_t - P_t^N\right) g(x)\right| \leq \int_0^t \left|P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x)\right| ds$$

Convergence of the semigroups (2)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

$$\begin{aligned}\left|\left(\bar{P}_t - P_t^N\right) g(x)\right| &\leq \int_0^t \left|P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x)\right| ds \\ &\leq \int_0^t \mathbb{E}_x \left[\left|\left(\bar{A} - A^N\right) \bar{P}_s g(X_{t-s}^N)\right|\right] ds\end{aligned}$$

Convergence of the semigroups (2)

$$\left(\bar{P}_t - P_t^N\right) g(x) = \int_0^t P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x) ds$$

$$\begin{aligned}\left|\left(\bar{P}_t - P_t^N\right) g(x)\right| &\leq \int_0^t \left|P_{t-s}^N \left(\bar{A} - A^N\right) \bar{P}_s g(x)\right| ds \\ &\leq \int_0^t \mathbb{E}_x \left[\left|\left(\bar{A} - A^N\right) \bar{P}_s g(X_{t-s}^N)\right|\right] ds \\ &\longrightarrow 0\end{aligned}$$

Convergence in finite-dimensional distribution

Convergence of the semigroups :

$$\mathbb{E}_x\left[g\left(X_t^N\right)\right] \longrightarrow \mathbb{E}_x\left[g\left(\bar{X}_t\right)\right]$$

Convergence in finite-dimensional distribution

Convergence of the semigroups :

$$\mathbb{E}_x\left[g\left(X_t^N\right)\right] \longrightarrow \mathbb{E}_x\left[g\left(\bar{X}_t\right)\right]$$

Induction + classical argument of Markov theory

$\Longrightarrow$ Convergence in finite-dimensional distribution :

$$\mathbb{E}_x\left[g_1\left(X_{t_1}^N\right)\dots g_n\left(X_{t_n}^N\right)\right] \longrightarrow \mathbb{E}_x\left[g_1\left(\bar{X}_{t_1}\right)\dots g_n\left(\bar{X}_{t_n}\right)\right]$$

Convergence of the processes

- X^N converges in fidi distribution to $\bar{X}$
- $\{X^N \ : \ N \in \mathbb{N}^*\}$ tight on $D(\mathbb{R}_+, \mathbb{R})$ (admited)

Convergence of the processes

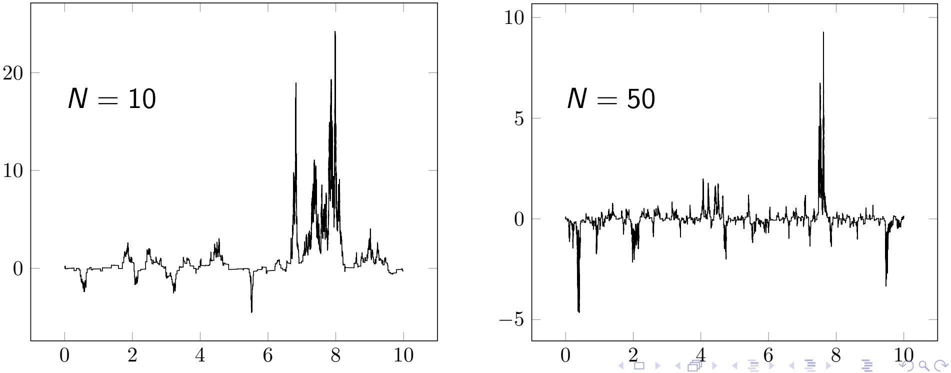
- X^N converges in fidi distribution to $\bar{X}$
- $\{X^N \ : \ N \in \mathbb{N}^*\}$ tight on $D(\mathbb{R}_+, \mathbb{R})$ (admited)

$$\Longrightarrow X^N \text{ converges to } \bar{X} \text{ in distribution in } D(\mathbb{R}_+, \mathbb{R})$$

Convergence of the processes

- X^N converges in fidi distribution to $\bar{X}$
- $\{X^N \ : \ N \in \mathbb{N}^*\}$ tight on $D(\mathbb{R}_+, \mathbb{R})$ (admited)

$$\implies X^N \text{ converges to } \bar{X} \text{ in distribution in } D(\mathbb{R}_+, \mathbb{R})$$



Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(X_{s-}^N)\}} d\pi_i^N(s,z)$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\le f(X_{s-}^N)\}} d\pi_i^N(s,z)$$

$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\le f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z)$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(X_{s-}^N)\}} d\pi_i^N(s,z) = \Phi(X^N, \pi_i^N)(t)$$

$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X}, \bar{\pi}_i)(t)$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z \leq f(X_{s-}^N)\}} d\pi_i^N(s,z) = \Phi(X^N, \pi_i^N)(t)$$

$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z \leq f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X}, \bar{\pi}_i)(t)$$

Φ continuous in $(\bar{X}, \bar{\pi}_i)$ as

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(X^N_{s-})\}} d\pi_i^N(s,z) = \Phi(X^N, \pi_i^N)(t)$$

$$\bar{Z}_t^{i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X}, \bar{\pi}_i)(t)$$

Φ continuous in $(\bar{X}, \bar{\pi}_i)$ as
Skorohod's Representation Theorem :

$$(X^N, \pi_i^N) \xrightarrow{\mathcal{L}} (\bar{X}, \bar{\pi}_i)$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(X^N_{s-})\}} d\pi_i^N(s,z) = \Phi(X^N, \pi_i^N)(t)$$

$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X}, \bar{\pi}_i)(t)$$

Φ continuous in $(\bar{X}, \bar{\pi}_i)$ as
Skorohod's Representation Theorem :

$$\begin{array}{ccc} (X^N, \pi_i^N) & \xrightarrow{\mathcal{L}} & (\bar{X}, \bar{\pi}_i) \\ \mathcal{L} \ \Big\| & & \Big\| \ \mathcal{L} \\ (\widetilde{X}^N, \widetilde{\pi}^N) & \xrightarrow{as} & (\widetilde{X}, \widetilde{\pi}) \end{array}$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(X_{s-}^N)\}} d\pi_i^N(s,z) = \Phi(X^N,\pi_i^N)(t)$$

$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\leq f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X},\bar{\pi}_i)(t)$$

Φ continuous in $(\bar{X},\bar{\pi}_i)$ as
Skorohod's Representation Theorem :

$$\begin{array}{ccc} (X^N,\pi_i^N) & \xrightarrow{\mathcal{L}} & (\bar{X},\bar{\pi}_i) \\ \mathcal{L}\ \| & & \|\ \mathcal{L} \\ (\widetilde{X}^N,\widetilde{\pi}^N) & \xrightarrow{as} & (\widetilde{X},\widetilde{\pi}) \end{array} \implies \Phi(\widetilde{X}^N,\widetilde{\pi}^N) \xrightarrow{as} \Phi(\widetilde{X},\widetilde{\pi})$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\le f(X^N_{s-})\}} d\pi_i^N(s,z) = \Phi(X^N,\pi_i^N)(t)$$

$$\bar{Z}_t^{i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\le f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X},\bar{\pi}_i)(t)$$

Φ continuous in $(\bar{X},\bar{\pi}_i)$ as
Skorohod's Representation Theorem :

$$\begin{array}{ccccccc} (X^N,\pi_i^N) & \xrightarrow{\mathcal{L}} & (\bar{X},\bar{\pi}_i) & & \Phi(X^N,\pi_i^N) & \xrightarrow{\mathcal{L}} & \Phi(\bar{X},\bar{\pi}_i) \\ \mathcal{L}\ \| & & \|\ \mathcal{L} & \Longrightarrow & \mathcal{L}\ \| & & \|\ \mathcal{L} \\ (\widetilde{X}^N,\widetilde{\pi}^N) & \xrightarrow{as} & (\widetilde{X},\widetilde{\pi}) & & \Phi(\widetilde{X}^N,\widetilde{\pi}^N) & \xrightarrow{as} & \Phi(\widetilde{X},\widetilde{\pi}) \end{array}$$

Convergence of $Z^{N,i}$

$$Z_t^{N,i} := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\le f(X^N_{s-})\}} d\pi_i^N(s,z) = \Phi(X^N,\pi_i^N)(t)$$

$$\bar{Z}_t^i := \int_{]0,t]\times\mathbb{R}_+} 1_{\{z\le f(\bar{X}_{s-})\}} d\bar{\pi}_i(s,z) = \Phi(\bar{X},\bar{\pi}_i)(t)$$

Φ continuous in $(\bar{X},\bar{\pi}_i)$ as
Skorohod's Representation Theorem :

$$\begin{array}{ccccccc}
(X^N,\pi_i^N) & \xrightarrow{\mathcal{L}} & (\bar{X},\bar{\pi}_i) & & \Phi(X^N,\pi_i^N) & \xrightarrow{\mathcal{L}} & \Phi(\bar{X},\bar{\pi}_i) \\
\mathcal{L}\ \| & & \|\ \mathcal{L} & \Longrightarrow & \mathcal{L}\ \| & & \|\ \mathcal{L} \\
(\widetilde{X}^N,\widetilde{\pi}^N) & \xrightarrow{as} & (\widetilde{X},\widetilde{\pi}) & & \Phi(\widetilde{X}^N,\widetilde{\pi}^N) & \xrightarrow{as} & \Phi(\widetilde{X},\widetilde{\pi})
\end{array}$$

Result : $(Z^{N,i})_{i\ge1}$ converges to $(\bar{Z}^i)_{i\ge1}$ in distribution in $D(\mathbb{R}_+,\mathbb{R})^{\mathbb{N}^*}$

Bibliography

- Erny, Löcherbach, Loukianova (2019). Mean field limits for interacting Hawkes processes in a diffusive regime. HAL, arXiv.
- Brémaud, Massoulié (1996). Stability of nonlinear Hawkes processes. Annals of probability.
- Daley, Vere-Jones (2003). An Introduction to the Theory of Point Processes : Volume I : Elementary Theory and Methods. Springer.

Thank you for your attention !

Questions ?