

On the mixing time of the flip walk on triangulations of the sphere

Thomas Budzinski

ENS Paris

Les probabilités de demain

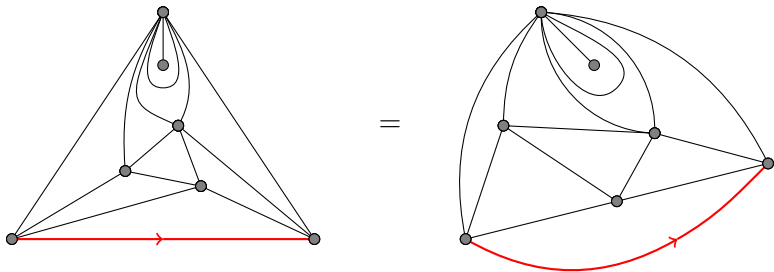
11 Mai 2017

Definitions

- A *planar map* is a finite, connected graph embedded in the sphere in such a way that no two edges cross (except at a common endpoint), considered up to orientation-preserving homeomorphism.
- A planar map is a *rooted type-1 triangulation* if all its faces have degree 3 and it has a distinguished oriented edge. It may contain multiple edges and loops.

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Random planar maps in a nutshell

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- the Brownian map is homeomorphic to the sphere [Le Gall–Paulin].

A uniform triangulation of the sphere with 10 000 vertices

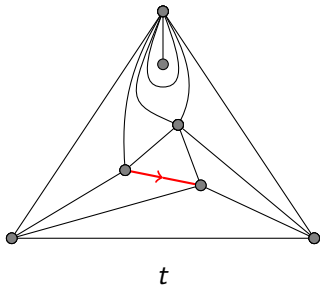


How to sample a large uniform triangulation ?

- "Modern" tools : bijections with trees, peeling process.
- Back in the 80's : Monte Carlo methods : we look for a Markov chain on \mathcal{T}_n for which the uniform measure is stationary.
- A simple local operation on triangulations : flips.

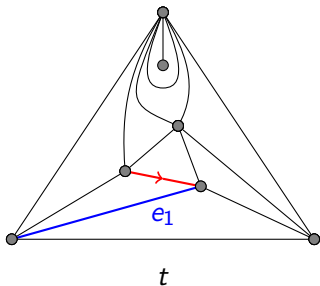
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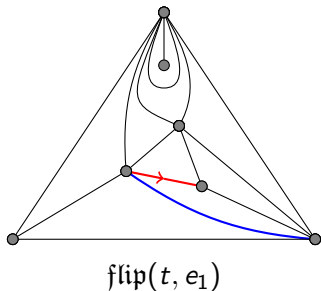


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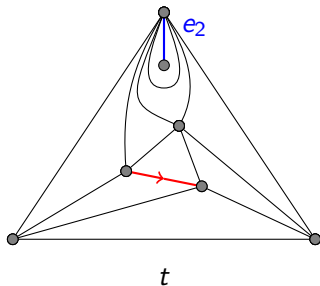
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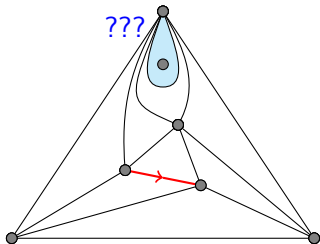
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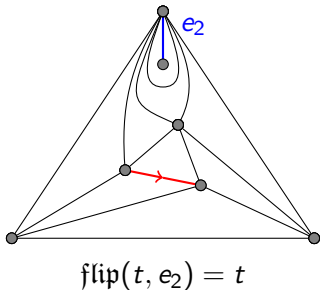
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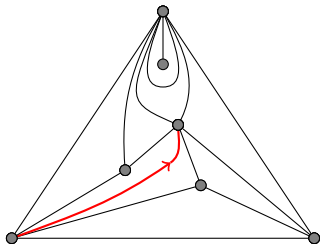
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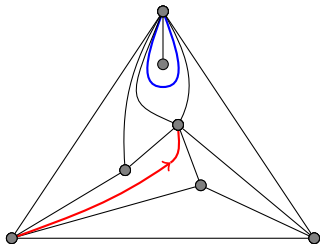
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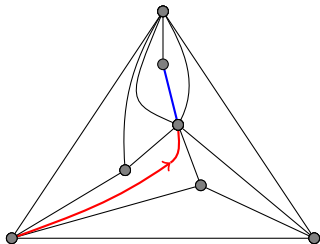
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A Markov chain on \mathcal{T}_n

- We fix $t_0 \in \mathcal{T}_n$ and take $T_n(0) = t_0$.
- Conditionally on $(T_n(k))_{0 \leq i \leq k}$, let e_k be a uniform edge of $T_n(k)$ and $T_n(k+1) = \text{flip}(T_n(k), e_k)$.

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- The chain T_n is irreducible (the flip graph is connected [Wagner 36]) and aperiodic (non flippable edges), so it converges to the uniform measure.
- Question : how quick is the convergence ?

- For $n \geq 3$ and $0 < \varepsilon < 1$ we define the mixing time $t_{mix}(\varepsilon, n)$ as the smallest k such that

$$\max_{t_0 \in T_n} \max_{A \subset T_n} |\mathbb{P}(T_n(k) \in A) - \mathbb{P}(T_n(\infty) \in A)| \leq \varepsilon,$$

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Theorem (B., 2016)

For all $0 < \varepsilon < 1$, there is a constant $c > 0$ such that

$$t_{mix}(\varepsilon, n) \geq cn^{5/4}.$$

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Theorem (\approx Le Gall–Paulin, 2008)

Let $\ell_n = o(n^{1/4})$. Then, with probability going to 1 as $n \rightarrow +\infty$, there is no cycle in $T_n(\infty)$ of length at most ℓ_n that separates $T_n(\infty)$ in two parts, each of which contains at least $\frac{n}{4}$ vertices.

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Let $T_n^1(0)$ and $T_n^2(0)$ be two independent uniform triangulations of a 1-gon with $\frac{n}{2}$ inner vertices each, and $T_n(0)$ the gluing of $T_n^1(0)$ and $T_n^2(0)$ along their boundary.

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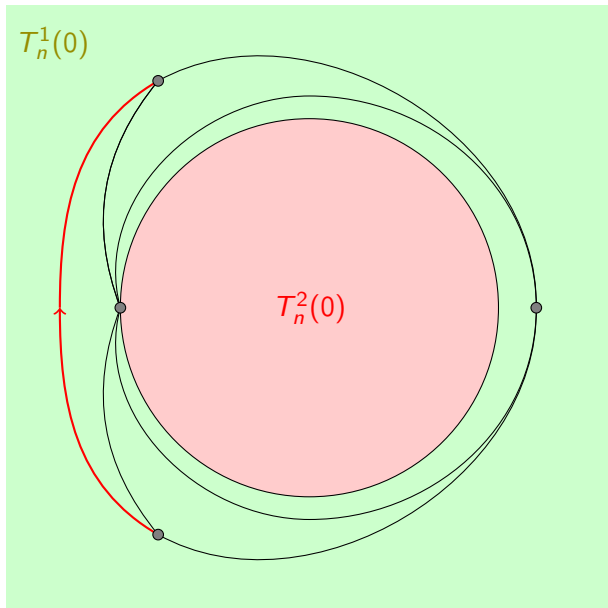
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Proposition

Let $k_n = o(n^{5/4})$. There is a cycle γ in $T_n(k_n)$ of length $o(n^{1/4})$ in probability that separates $T_n(k_n)$ in two parts, each of which contains at least $\frac{n}{4}$ vertices.

Exploration of $T_n(k)$



Perimeter :

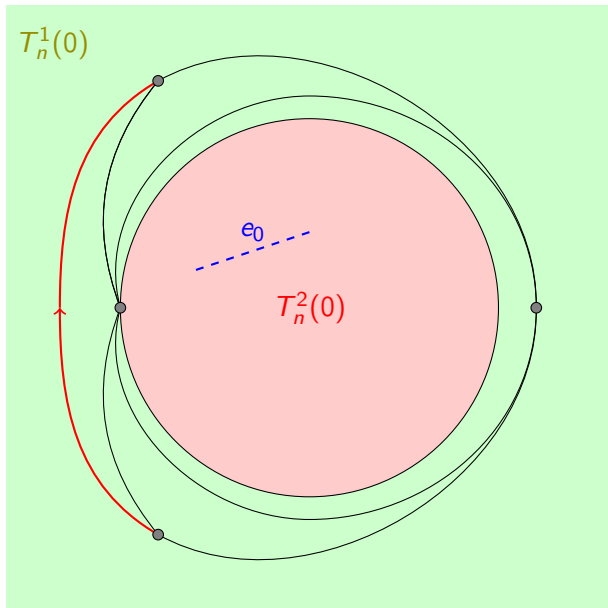
$$\tilde{P}_n(0) = 1$$

Explored volume :

$$\tilde{V}_n(0) = 1$$

exploration steps :

Exploration of $T_n(k)$



Perimeter :

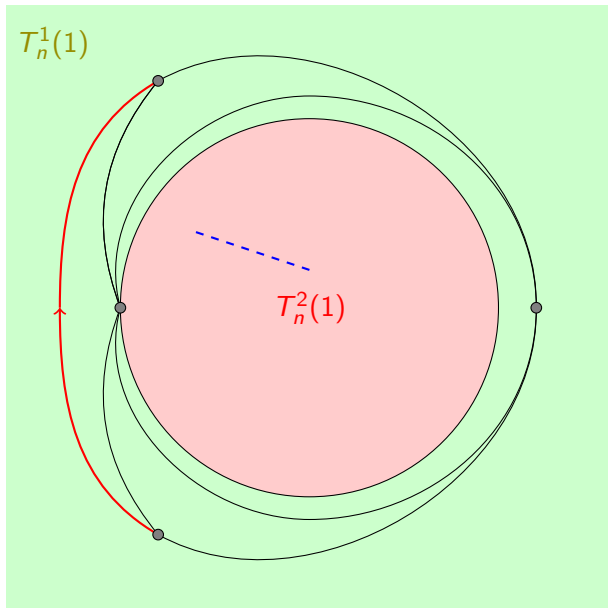
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Perimeter :

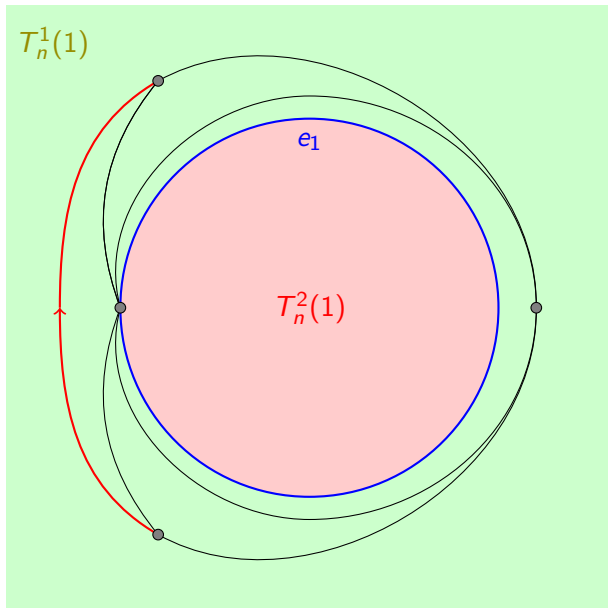
$$\tilde{P}_n(1) = 1$$

Explored volume :

$$\tilde{V}_n(1) = 1$$

exploration steps :

Exploration of $T_n(k)$



Perimeter :

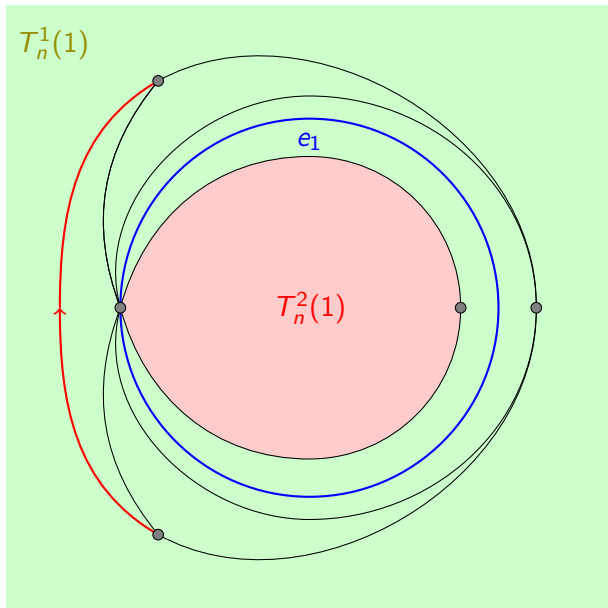
$$\tilde{P}_n(1) = 1$$

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exploration steps :

Exploration of $T_n(k)$



Perimeter :

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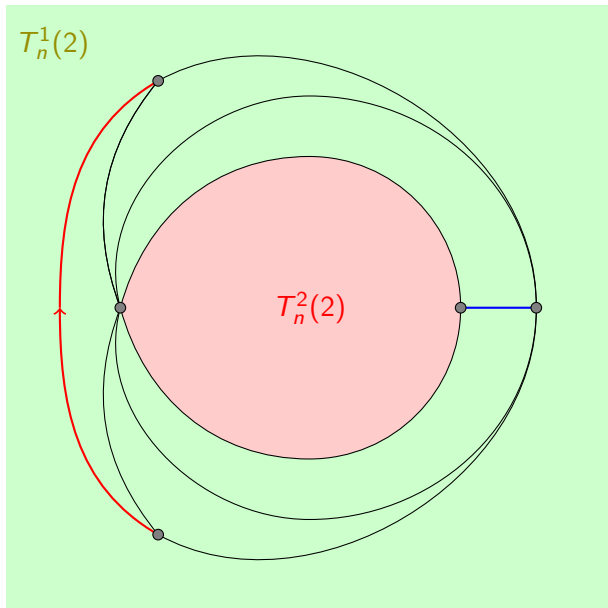
Explored volume :

$$\tilde{V}_n(1) = 1$$

exploration steps :

1

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(2) = 2$$

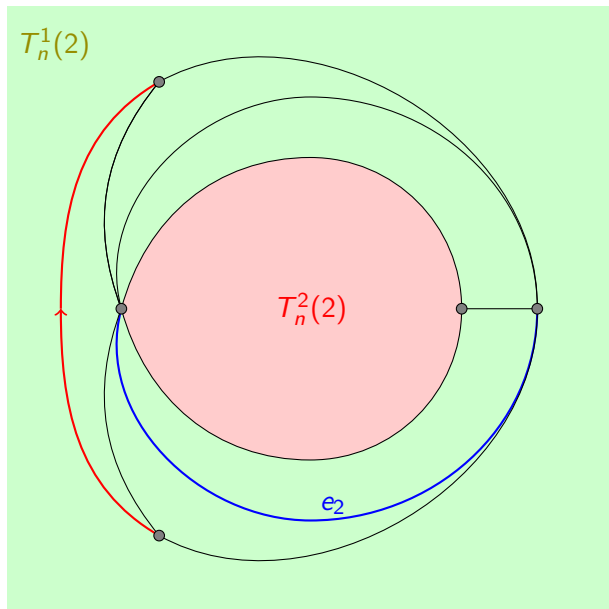
Explored volume :

$$\tilde{V}_n(2) = 2$$

exploration steps :

1

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(2) = 2$$

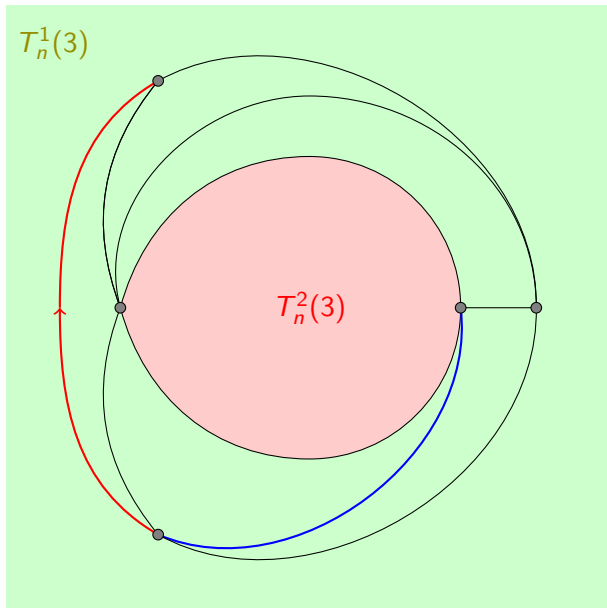
Explored volume :

$$\tilde{V}_n(2) = 2$$

exploration steps :

1

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(3) = 2$$

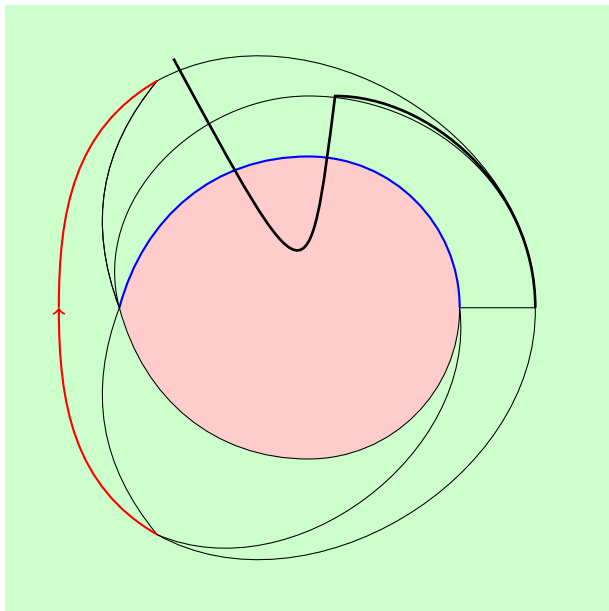
Explored volume :

$$\tilde{V}_n(3) = 2$$

exploration steps :

1

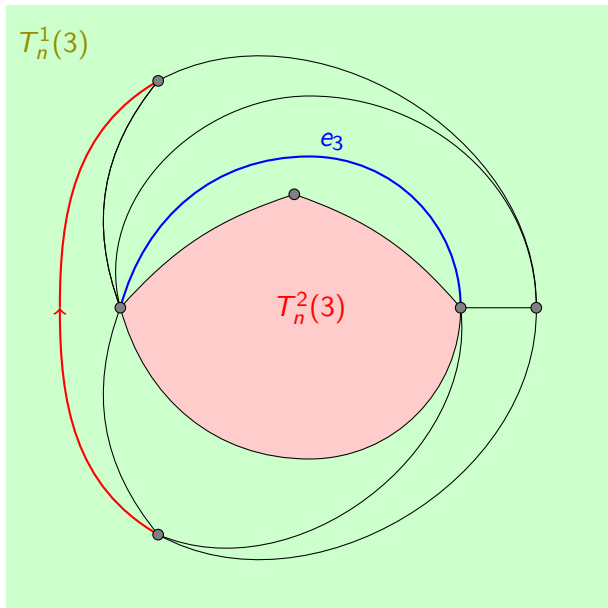
Exploration of $T_n(k)$



exploration steps :

1

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(3) = 2$$

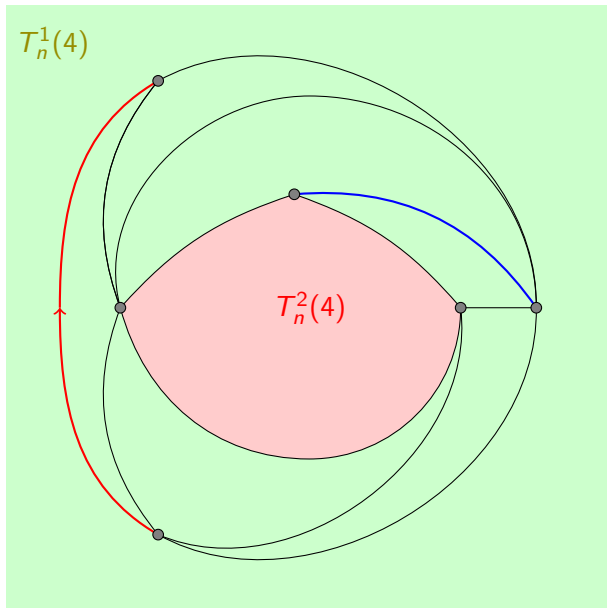
Explored volume :

$$\tilde{V}_n(3) = 2$$

exploration steps :

1
3

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(4) = 3$$

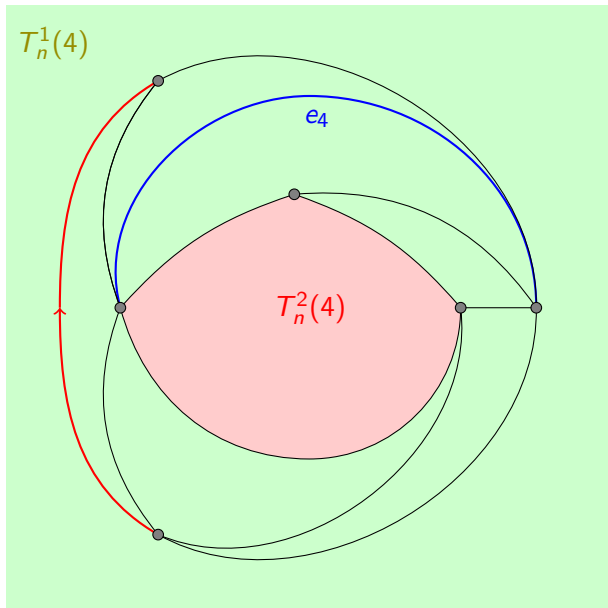
Explored volume :

$$\tilde{V}_n(4) = 3$$

exploration steps :

1
3

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(4) = 3$$

Explored volume :

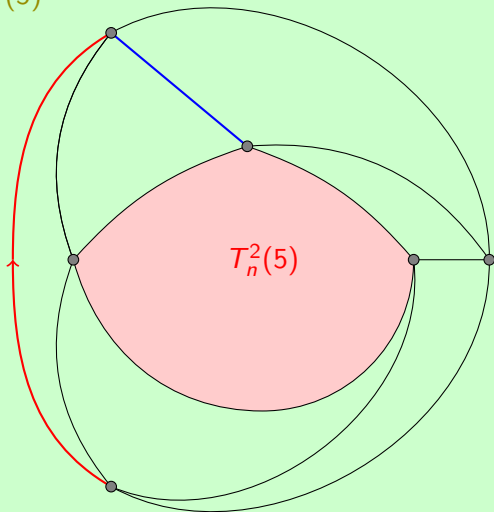
$$\tilde{V}_n(4) = 3$$

exploration steps :

1
3

Exploration of $T_n(k)$

$T_n^1(5)$



Perimeter :

$$\tilde{P}_n(5) = 3$$

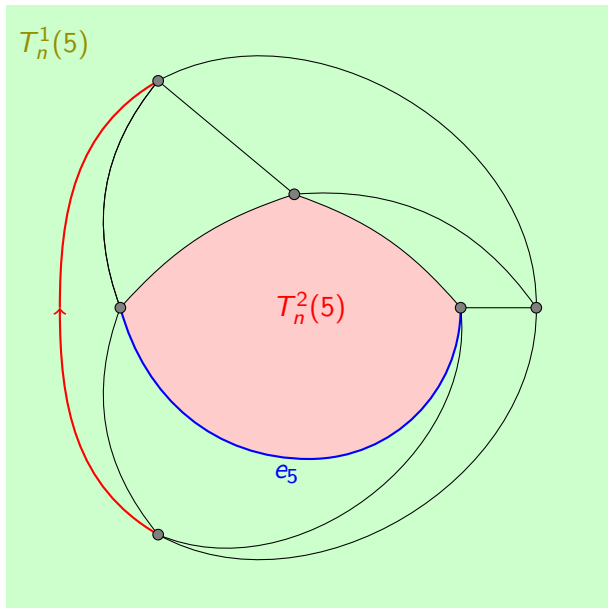
Explored volume :

$$\tilde{V}_n(5) = 3$$

exploration steps :

1
3

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(5) = 3$$

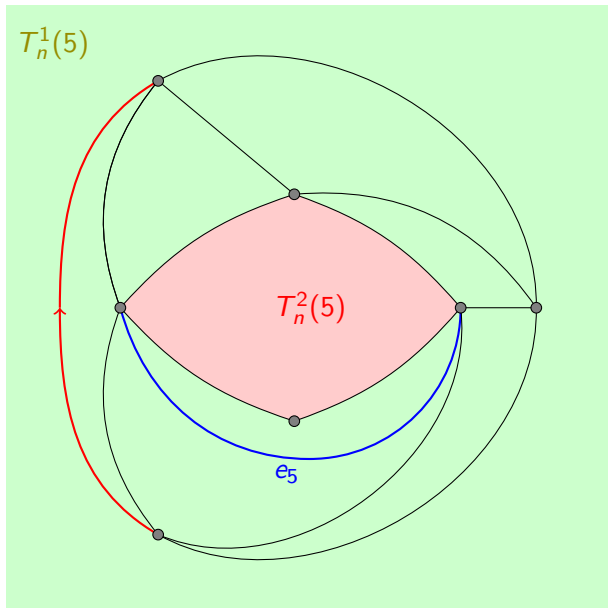
Explored volume :

$$\tilde{V}_n(5) = 3$$

exploration steps :

1
3

Exploration of $T_n(k)$



Perimeter :

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Explored volume :

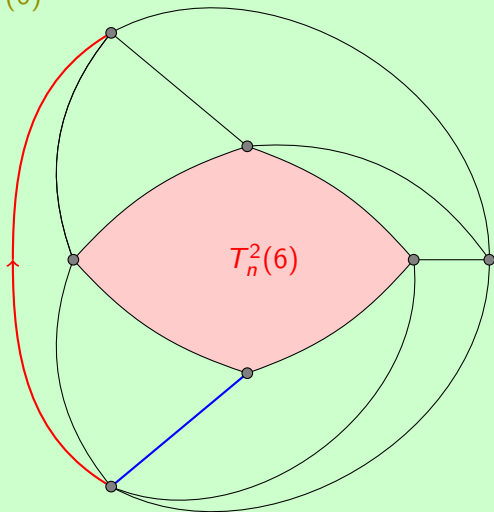
$$\tilde{V}_n(5) = 3$$

exploration steps :

1
3
5

Exploration of $T_n(k)$

$T_n^1(6)$



Perimeter :

$$\tilde{P}_n(6) = 4$$

Explored volume :

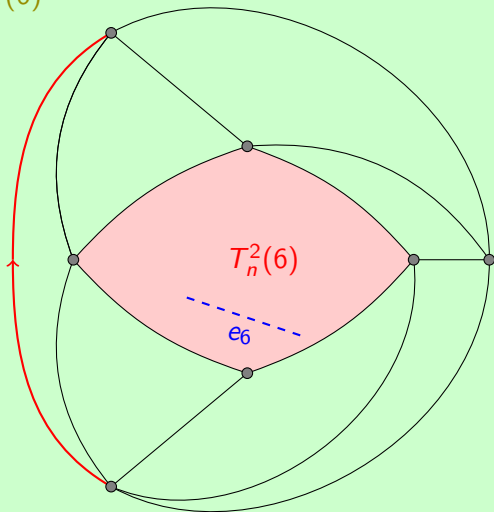
$$\tilde{V}_n(6) = 4$$

exploration steps :

1
3
5

Exploration of $T_n(k)$

$T_n^1(6)$



Perimeter :

$$\tilde{P}_n(6) = 4$$

Explored volume :

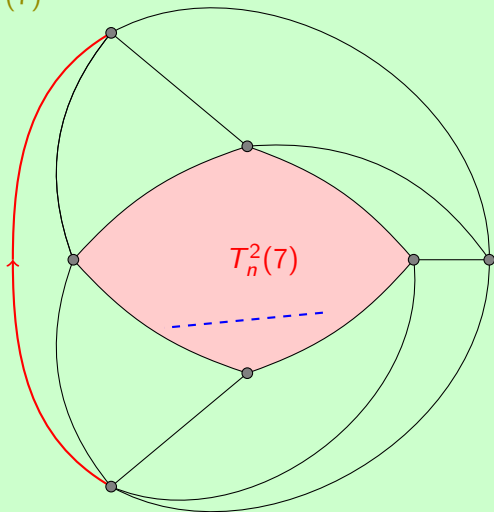
$$\tilde{V}_n(6) = 4$$

exploration steps :

1
3
5

Exploration of $T_n(k)$

$T_n^1(7)$



Perimeter :

$$\tilde{P}_n(7) = 4$$

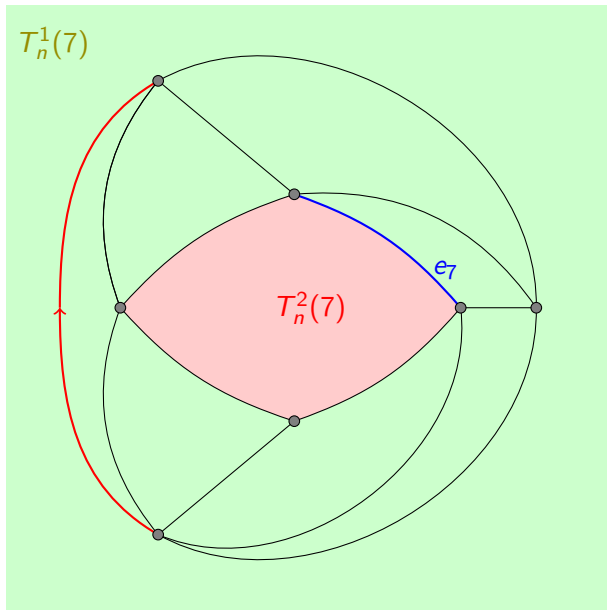
Explored volume :

$$\tilde{V}_n(7) = 4$$

exploration steps :

1
3
5

Exploration of $T_n(k)$

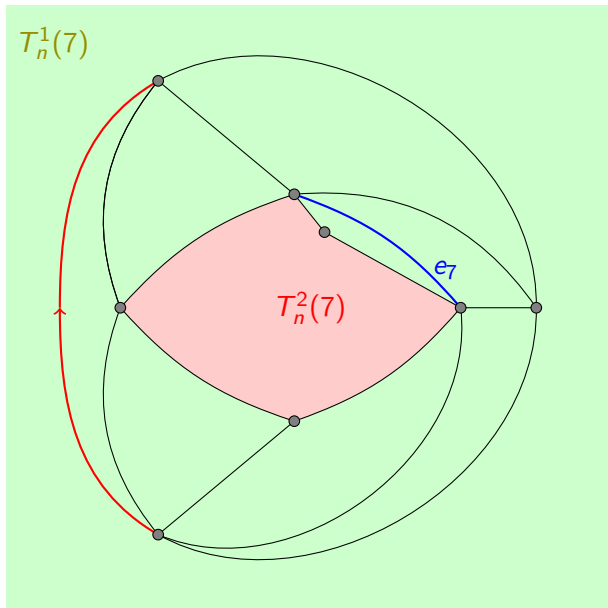


Perimeter $P_n(0) = -225$

exploration steps :

1
3
5

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(7) = 4$$

Explored volume :

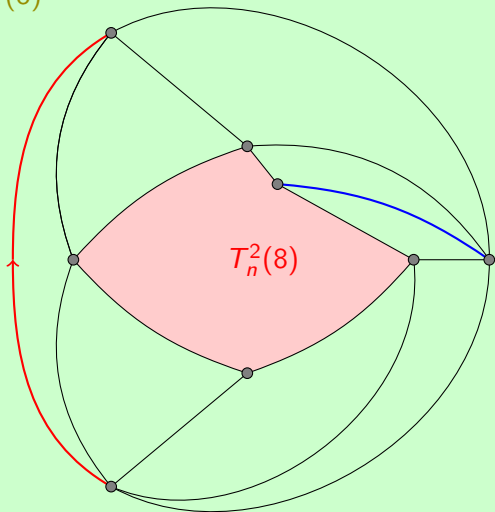
$$\tilde{V}_n(7) = 4$$

exploration steps :

1
3
5
7

Exploration of $T_n(k)$

$T_n^1(8)$



Perimeter :

$$\tilde{P}_n(8) = 5$$

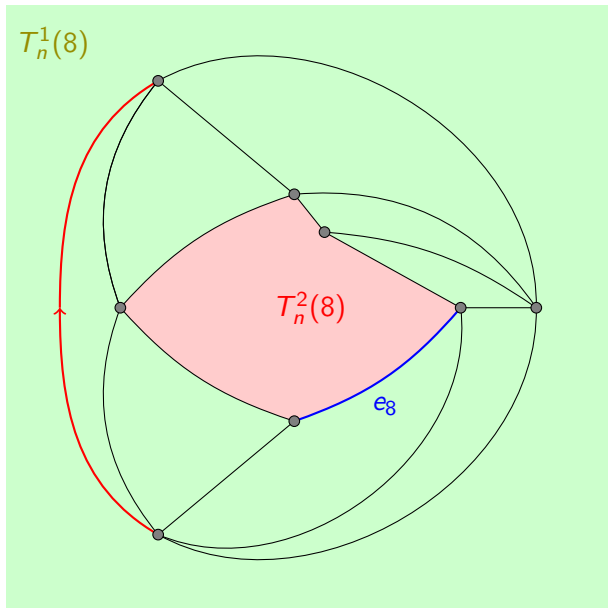
Explored volume :

$$\tilde{V}_n(8) = 5$$

exploration steps :

1
3
5
7

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(8) = 5$$

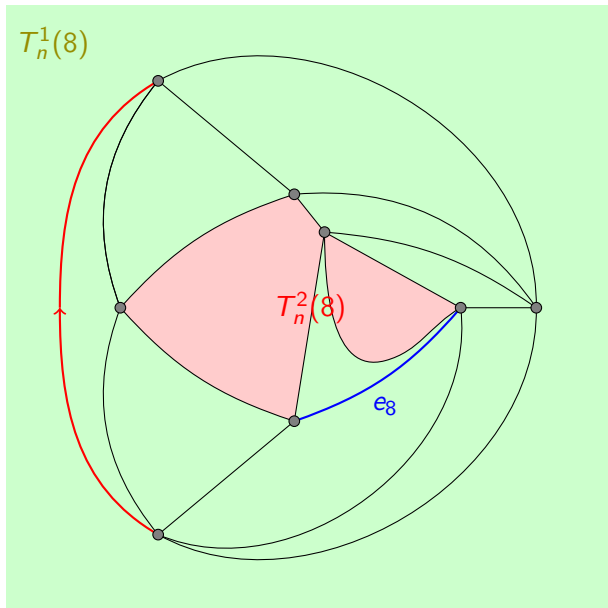
Explored volume :

$$\tilde{V}_n(8) = 5$$

exploration steps :

1
3
5
7

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(8) = 5$$

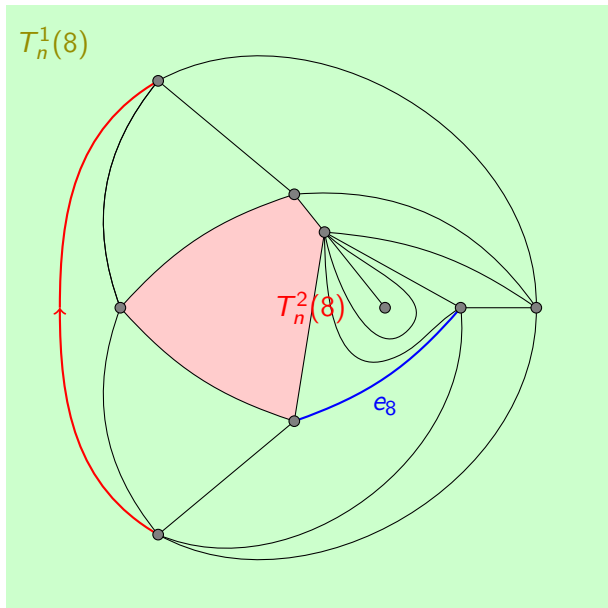
Explored volume :

$$\tilde{V}_n(8) = 5$$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$



Perimeter :

$$\tilde{P}_n(8) = 5$$

Explored volume :

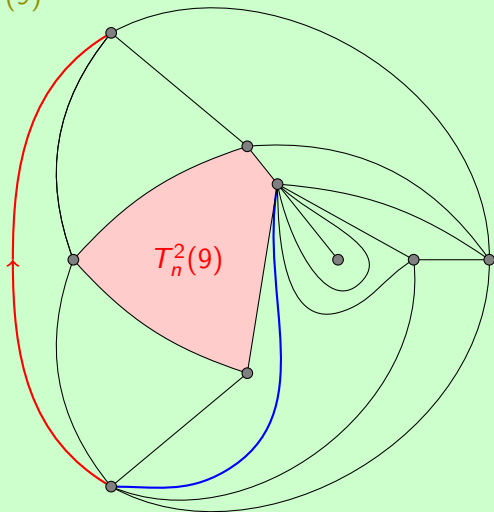
$$\tilde{V}_n(8) = 5$$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

$T_n^1(9)$



Perimeter :

$$\tilde{P}_n(9) = 4$$

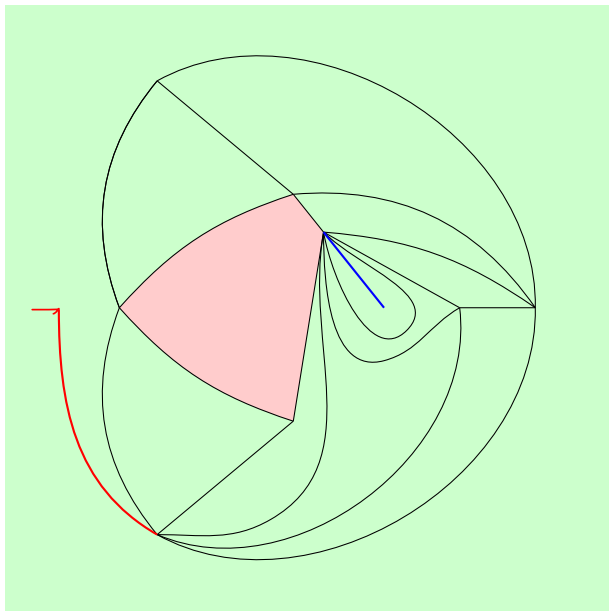
Explored volume :

$$\tilde{V}_n(9) = 6$$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

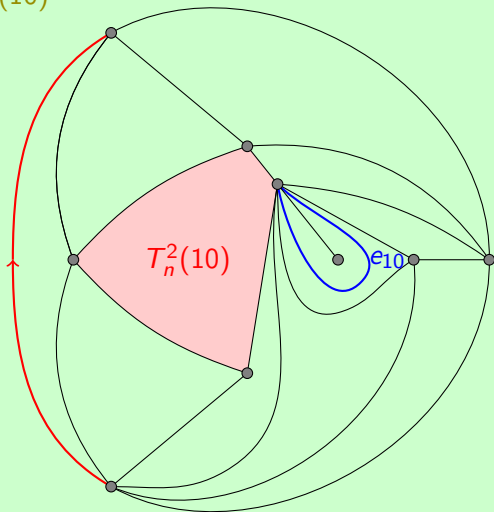


exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

$T_n^1(10)$



Perimeter :

$$\tilde{P}_n(10) = 4$$

Explored volume :

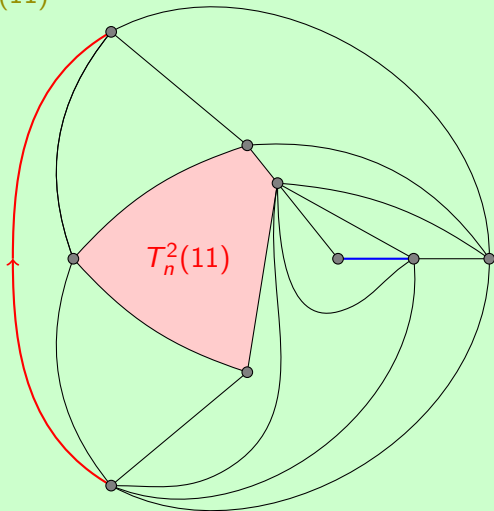
$$\tilde{V}_n(10) = 6$$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

$T_n^1(11)$



Perimeter :

$$\tilde{P}_n(11) = 4$$

Explored volume :

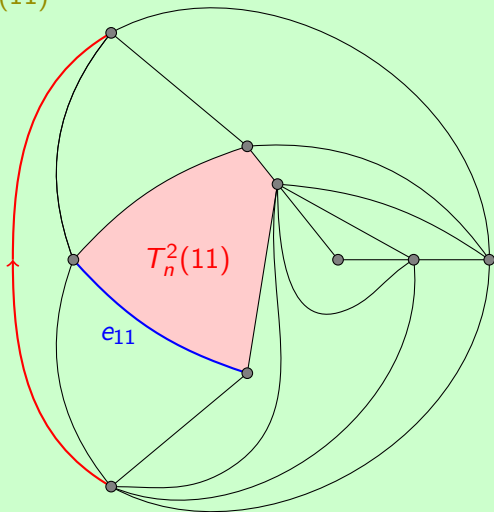
$$\tilde{V}_n(11) = 6$$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

$T_n^1(11)$



Perimeter :

$$\tilde{P}_n(11) = 4$$

Explored volume :

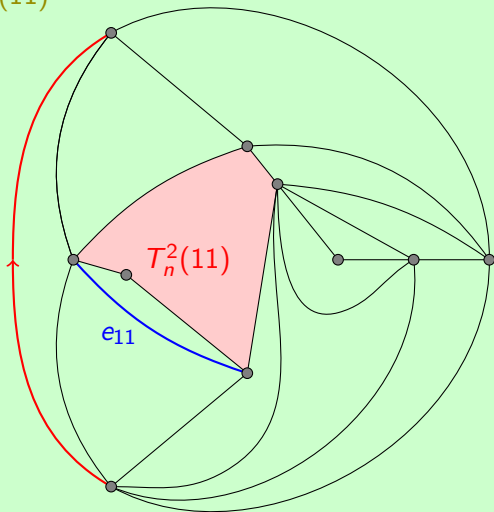
$$\tilde{V}_n(11) = 6$$

exploration steps :

1
3
5
7
8

Exploration of $T_n(k)$

$T_n^1(11)$



Perimeter :

$$\tilde{P}_n(11) = 4$$

Explored volume :

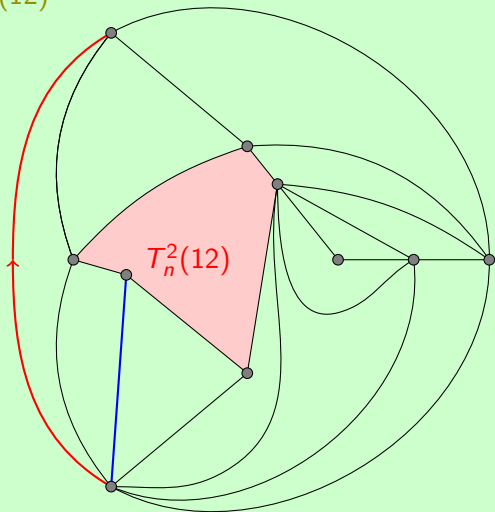
$$\tilde{V}_n(11) = 6$$

exploration steps :

1
3
5
7
8
11

Exploration of $T_n(k)$

$T_n^1(12)$



Perimeter :

$$\tilde{P}_n(12) = 5$$

Explored volume :

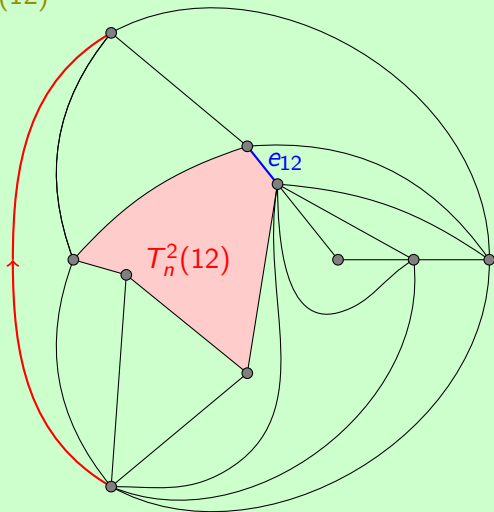
$$\tilde{V}_n(12) = 7$$

exploration steps :

1
3
5
7
8
11

Exploration of $T_n(k)$

$T_n^1(12)$



Perimeter :

$$\tilde{P}_n(12) = 5$$

Explored volume :

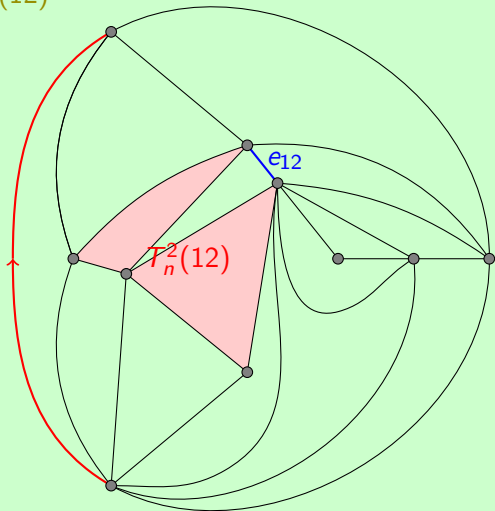
$$\tilde{V}_n(12) = 7$$

exploration steps :

1
3
5
7
8
11

Exploration of $T_n(k)$

$T_n^1(12)$



Perimeter :

$$\tilde{P}_n(12) = 5$$

Explored volume :

$$\tilde{V}_n(12) = 7$$

exploration steps :

1

3

5

7

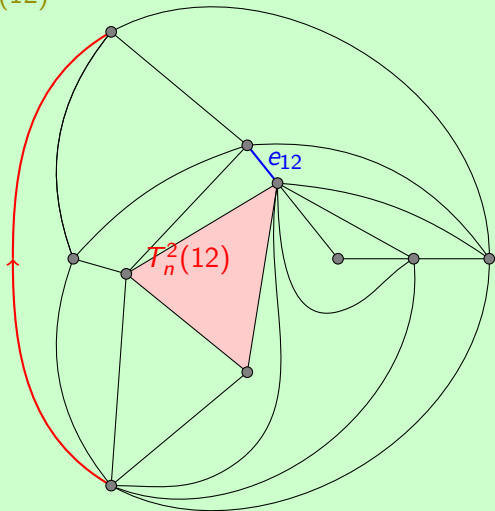
8

11

12

Exploration of $T_n(k)$

$T_n^1(12)$



Perimeter :

$$\tilde{P}_n(12) = 5$$

Explored volume :

$$\tilde{V}_n(12) = 7$$

exploration steps :

1

3

5

7

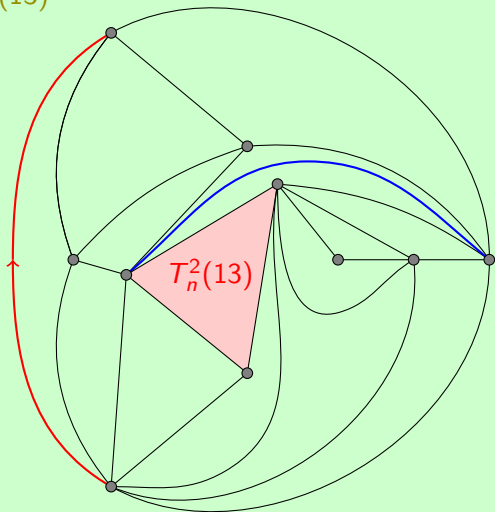
8

11

12

Exploration of $T_n(k)$

$T_n^1(13)$



Perimeter :

$$\tilde{P}_n(13) = 3$$

Explored volume :

$$\tilde{V}_n(13) = 7$$

exploration steps :

1

3

5

7

8

11

12

Peeling estimates

Let $P_n(j)$ and $V_n(j)$ be the perimeter and the explored volume after j **exploration steps**.

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$(P_n(j)/n^{2/3})$ and $(V_n(j)/n^{4/3})$ distributions converge to fixed, universal [Cormen-Le Gall] regulation.

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- So we need $\approx n^{5/4}$ flips to perform $\approx n^{3/4}$ exploration steps.
- In a uniform triangulation with perimeter p , we can find a cycle close to the boundary of length $\approx \sqrt{p}$ [Krikun].
- After $o(n^{5/4})$ flips, the perimeter is $o(\sqrt{n})$, so there is a separating cycle of length $o(n^{1/4})$.

Is the lower bound sharp?

- Back-of-the-envelope computation :
 - in a typical triangulation, the distance between two typical vertices x and y is $\approx n^{1/4}$.
 - The probability that a flip hits a geodesic is $\approx n^{-3/4}$.
 - The distance between x and y changes $\approx kn^{-3/4}$ times before time k .
 - If $d(x, y)$ evolves roughly like a random walk, it varies of $\approx \sqrt{kn^{-3/4}} = n^{1/4}$ for $k = n^{5/4}$.

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- For triangulations of a convex polygon (no inner vertices), the lower bound $n^{3/2}$ is believed to be sharp but the best known upper bound is n^5 [McShine–Tetali].
- Prove that the mixing time is polynomial?

MERCI!