Homogenization on supercritical percolation cluster

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The model

Model: percolation on the hypercubic lattice

- lattice $\mathbb{Z}^d$, $d \geq 2$.
- V set of vertices: $V := \mathbb{Z}^d$.
- E_d set of edges: $E_d := \left\{ (x, y) \; : \; x, y \in \mathbb{Z}^d, \, |x - y|_1 = 1 \right\}$.

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- E_d set of edges: $E_d := \left\{ (x, y) \ : \ x, y \in \mathbb{Z}^d, |x - y|_1 = 1 \right\}$.
- Fix $p \in [0, 1]$.
- $(\omega_e)_{e \in E_d}$ be a sequence of i.i.d Bernouilli random variables s.t

$$\mathbb{P}(\omega_e = 1) = 1 - \mathbb{P}(\omega_e = 0) = p.$$

Percolation phases

There exists $p_c = p_c(d) \in (0,1)$ such that

- $p < p_c$, subcritical phase
- $p = p_c$, critical phase
- $p > p_c$, supercritical phase $\rightarrow$ there exists a unique infinite cluster.

The supercritical percolation cluster is well-behaved

Theorem (Grimmett-Marstrand, 1990, Chayes-Chayes-Newman 1987)

- Assume $d \geq 3$ and let $p > p_c$, there exists $L := L(p, d) < \infty$ such that

$$\mathcal{C}_\infty \cap \left\{ x \in \mathbb{Z}^d \ : \ 0 \leq x_1 \leq L \right\}$$

  contains an infinite connected component of open edges almost surely.
- There exists $\xi(p) > 0$ such that, for each $x \in \mathbb{Z}^d$,

$$\mathbb{P}\left(0 \leftrightarrow x, \, 0 \not\leftrightarrow \infty\right) \leq \exp\left(-\xi(p)|x|\right).$$

The supercritical percolation cluster is well-behaved

Theorem (Antal-Pisztora, 1996)

Let $p > p_c$. There exists $\rho := \rho(p, d) < \infty$ and $\alpha := \alpha(d, p) > 0$ such that for each $y \in \mathbb{Z}^d$

$$\mathbb{P}\left(0 \leftrightarrow y, \operatorname{dist}(0, y) \geq \rho|y|\right) \leq \exp\left(-\alpha|y|\right).$$

A notion of good cube

Definition

We say that a cube $\square$ is decent if

- There exists a unique crossing cluster in $\square$, denoted by $\mathcal{C}(\square)$.
- All open paths of size larger than $\frac{\text{size}(\square)}{10}$ is connected to $\mathcal{C}(\square)$ within $\square$.

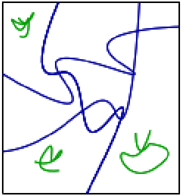

Figure 1: A decent box

A notion of good cube

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We say that a cube is good if $\square$ and $\frac{5}{4}\square$ are decent.

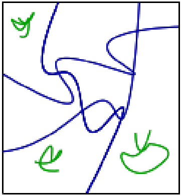

Figure 1: A decent box

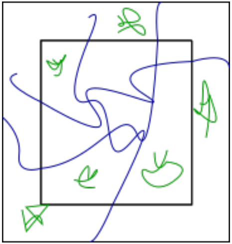

Figure 2: A good box

A notion of good cubes

Theorem (Penrose-Pisztora, 1996)

Let $\square$ be a cube in $\mathbb{Z}^d$, then there exists a constant $C := C(d,p) < \infty$,

$$\mathbb{P}\left(\square \text{ is a good cube}\right) \geq 1 - C\exp\left(-C^{-1}\,\text{size}\left(\square\right)\right).$$

A notion of good cubes

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Proposition

Given two cubes $\square_1$ and $\square_2$ with the same size and 1 face in common then $\mathcal{C}(\square_1)$ and $\mathcal{C}(\square_2)$ are connected within $\square_1 \cup \square_2$.

Proof of the proposition

By contradiction, assume that

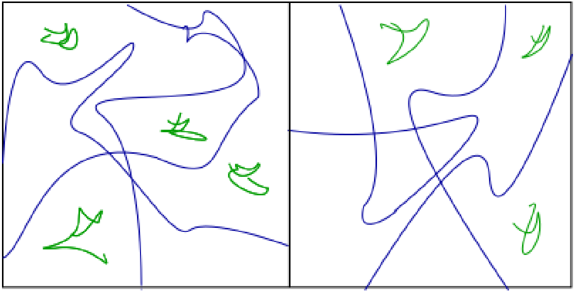

Figure 3: Two good boxes disconnected

Proof of the proposition

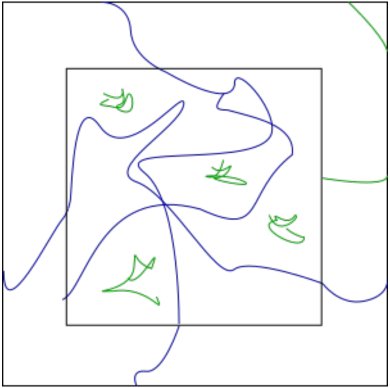

Figure 4: A look at $\frac{5}{4}\square_1 \ldots$

Proof of the proposition

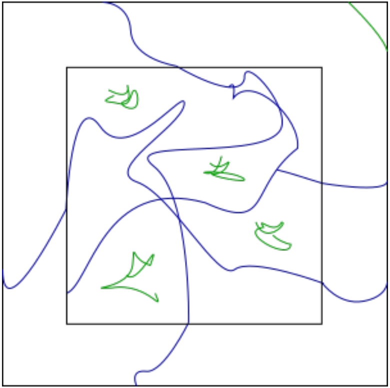

Figure 5: ...shows that the boxes are connected.

Proof of the proposition

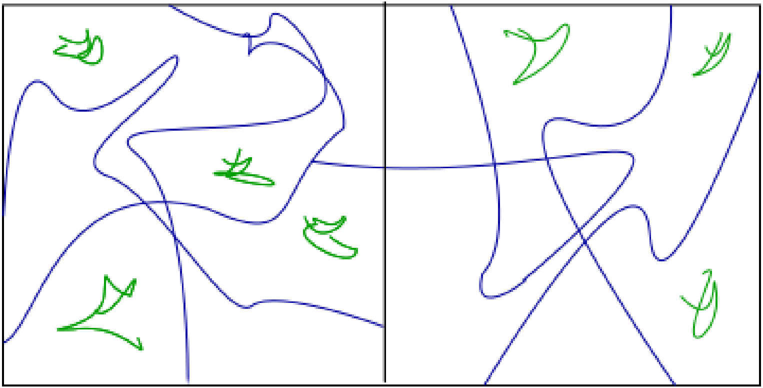

Figure 6: The bigger picture.

Triadic cubes

Definition

A triadic cube of $\mathbb{Z}^d$ is a cube of the form, for some $n \in \mathbb{N}$

$$z + \left(-\frac{3^n}{2}, \frac{3^n}{2} \right)^d, \; z \in 3^n \mathbb{Z}^d.$$

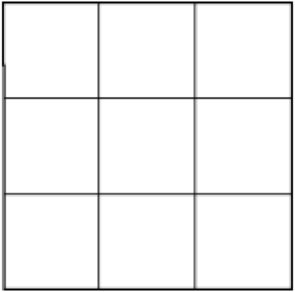

Figure 7: Triadic cubes.

Triadic cubes

Proposition

Two triadic cubes are either included in one another or disjoint.

- Idea: create of partition of good triadic cubes of different sizes.

A partition of good boxes

Proposition (A partition $\mathcal{P}$ of good boxes)

There exists, almost surely, a partition $\mathcal{P}$ of $\mathbb{Z}^d$ into good cubes such that

- Two neighboring cubes are of comparable sizes: for each $\square, \square' \in \mathcal{P}$ such that $\mathrm{dist}(\square, \square') \leq 1$,

$$\frac{1}{3} \leq \frac{\mathrm{size}(\square)}{\mathrm{size}(\square')} \leq 3.$$

- The size of each cube is "almost" bounded, for each $x \in \mathbb{Z}^d$,

$$\mathbb{E}\left[\exp\left(\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)\right)\right] < \infty.$$

What the partition looks like (without clusters)

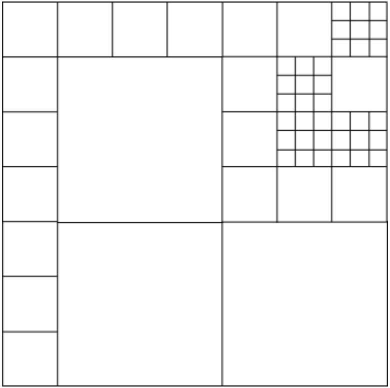

Figure 8: A partition of good cubes.

Application of the partition

- Estimating the chemical distance on the infinite cluster
- Extend a function defined on $\mathcal{C}_\infty$ to $\mathbb{Z}^d$
- Proving functional inequalities on the infinite cluster.

Thank you!