Random Matrices: Beyond Wigner and Marchenko-Pastur Laws

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Wigner's Matrices

Wishart's Matrices

Generalizations

Wigner's Matrices

$X_{ij}^{(n)}$, $(n, i, j \geq 1)$: collection of i.i.d. random variables such that

$$\mathbb{E}[X] = 0 \text{ and } \mathbb{E}[X^2] = 1.$$

For all $n \geq 1$:

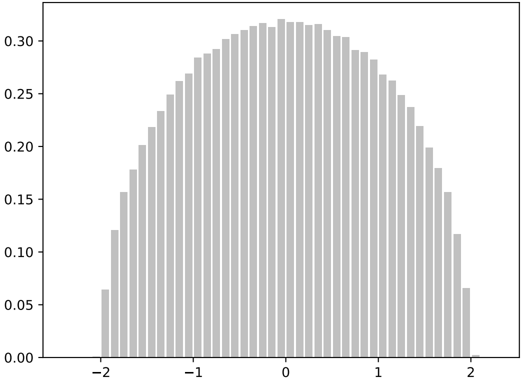
$$W_n = \frac{1}{\sqrt{n}} \begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} & \cdots & X_{1n}^{(n)} \\ X_{12}^{(n)} & X_{22}^{(n)} & \cdots & X_{2n}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1n}^{(n)} & X_{2n}^{(n)} & \cdots & X_{nn}^{(n)} \end{bmatrix}.$$

Symmetry: n real eigenvalues

$$\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_n^{(n)}.$$

Simulations

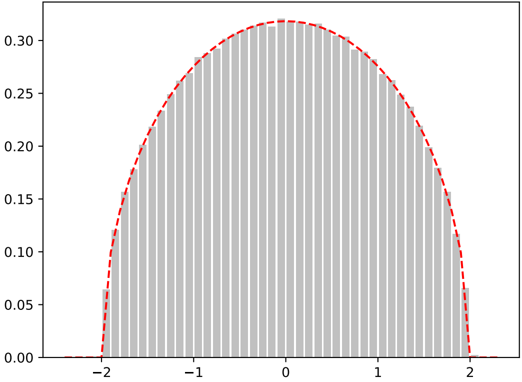
Gaussian case: $X \sim \mathcal{N}(0,1)$, 100 matrices of size 500×500.



Simulations

Gaussian case: $X \sim \mathcal{N}(0,1)$, 100 matrices of size 500×500.

$$\sigma(\mathrm{d}x) = \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}_{|x|\leq 2}\mathrm{d}x.$$



Convergence to the semi-circle law

Empirical spectral measure:

$$\mu_{W_n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^{(n)}}.$$

Theorem (Wigner, 1958)

Almost surely, μ_{W_n} *weakly converges to* σ*: for all bounded measurable function* $f : \mathbb{R} \mapsto \mathbb{R}_+$,

$$\int_{\mathbb{R}} f(x)\mathrm{d}\mu_{W_n}(x) = \frac{1}{n}\sum_{i=1}^{n} f\left(\lambda_i^{(n)}\right) \xrightarrow[n\to+\infty]{a.s.} \int_{-2}^{2} f(x)\frac{\sqrt{4-x^2}}{2\pi}\mathrm{d}x.$$

Wishart's Matrices

- $m = \alpha n$, $\alpha > 0$;
- for all $n \geq 1$, rectangular $n \times m$ matrix:

$$R_n = \begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} & \cdots & X_{1m}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} & \cdots & X_{2m}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1}^{(n)} & X_{n2}^{(n)} & \cdots & X_{nm}^{(n)} \end{bmatrix}.$$

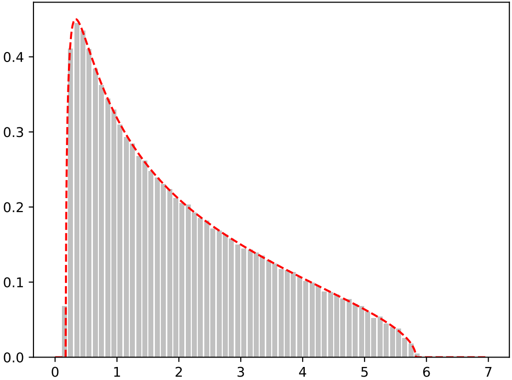
- Wishart's matrix of size n: covariance matrix

$$C_n = \frac{1}{n} R_n R_n^T.$$

- Positive matrix: n positive eigenvalues $\lambda_1^{(n)} \geq \cdots \geq \lambda_n^{(n)}$.

Simulations

Gaussian case: $X \sim \mathcal{N}(0,1)$, 100 matrices of size 500×1000 ($\alpha = 2$).

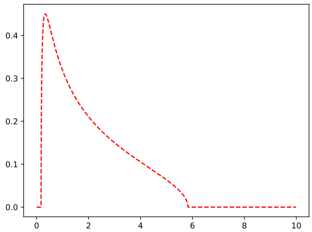


Convergence to the Marchenko-Pastur law

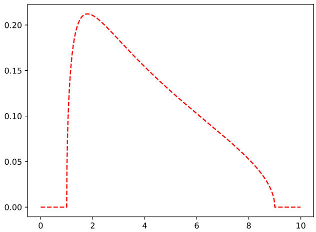
$$\mathrm{MP}_\alpha(\mathrm{d}x) = \frac{\sqrt{(b-x)(x-a)}}{2\pi x}\mathbf{1}_{(a,b)}(x)\mathrm{d}x + (1-\alpha)\mathbf{1}_{\alpha<1}\delta_0(\mathrm{d}x).$$

$$a = (1-\sqrt{\alpha})^2,\ b = (1+\sqrt{\alpha})^2.$$

$\alpha = 2$:



$\alpha = 4$:



Convergence to the Marchenko-Pastur law

Empirical spectral measure:

$$\mu_{C_n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^{(n)}}.$$

Theorem (Marchenko-Pastur, 1967)

Almost surely, μ_{W_n} weakly converges to MP_α: for all bounded measurable function $f : \mathbb{R} \mapsto \mathbb{R}_+$,

$$\int_{\mathbb{R}} f(x)\mathrm{d}\mu_{W_n}(x) = \frac{1}{n}\sum_{i=1}^{n} f\left(\lambda_i^{(n)}\right) \xrightarrow[n\to+\infty]{a.s.} \int_{\mathbb{R}} f(x)\mathrm{d}\mathrm{MP}_\alpha(x).$$

Wigner case: Generalization

$\{P_n\}$: sequence of probability laws, $M_2(P_n) = 1$.
Symmetric matrix:

$$W_n = \frac{1}{\sqrt{n}} \left(X_{ij}^{(n)} \right)_{1 \leq i,j \leq n},$$

$X_{ij}^{(n)}$'s i.d.d. with law P_n.

Convergence of spectral empirical measures μ_{W_n}:

- Ryan (1998);
- Zakharevich (2006).

Wishart case: Generalization

$\{P_n\}$: sequence of probability laws, $M_2(P_n) = 1$.

$$R_n = \left(X_{ij}^{(n)}\right) \in \mathcal{M}_{n\times m},$$

$m = \alpha n$; $X_{ij}^{(n)}$ i.d.d. with law P_n.

$$C_n = \frac{1}{n} R_n R_n^T.$$

Convergence of spectral empirical measures μ_{C_n}:

- Benaych-George and Cabanal-Duvillard (2012);
- N. (2017).

Theorem (N.,2017)

If

$$\frac{M_k(P_n)}{n^{k/2-1}} \underset{n\to+\infty}{\longrightarrow} A_k < \infty,$$

the weak limit of μ_{C_n} *only depends on* $(A_i)_i$ *and is characterized by its moments:*

$$m_k = \sum_{a=1}^{k}\sum_{l=1}^{a}\alpha^l \sum_{\mathbf{b}\in\mathcal{B}_{a,k}} |\mathcal{W}_k(a,l,\mathbf{b})| \prod_{i=1}^{a} A_{b_i}.$$

$\mathcal{W}_k(a,l,\mathbf{b})$: closed walks on planar rooted trees having a edges and l vertices in odd generations, with repetitions given by $\mathbf{b}$.

Combinatorial analysis of the formula

$$m_k = \sum_{l=1}^{k} \alpha^l |\mathcal{W}_k(k, l, (2, \dots, 2))| + A_4 \sum_{l=1}^{k-1} \alpha^l |\mathcal{W}_k(k-1, l, (4, 2, \dots, 2)) + \cdots$$

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$$m_k = m_k\left(\mathrm{MP}_\alpha\right) + A_4 m_k\left(\mathrm{MP}_\alpha^{\{1\}}\right) + \cdots$$

Combinatorial analysis of the formula

$$
\begin{aligned}
m_k &= \sum_{l=1}^{k} \alpha^l |\mathcal{W}_k(k,l,(2,\dots,2))| &+\quad A_4 \sum_{l=1}^{k-1} \alpha^l |\mathcal{W}_k(k-1,l,(4,2,\dots,2)) &+\cdots \\
m_k &= m_k\left(\mathrm{MP}_\alpha\right) &+\quad A_4 m_k\left(\mathrm{MP}_\alpha^{\{1\}}\right) &+\cdots
\end{aligned}
$$

$\mathrm{MP}_\alpha^{\{1\}}$: explicit signed measure with total mass zero.

$$
\mathrm{MP}_\alpha^{\{1\}}(\mathrm{d}x) = \frac{x^2 - 2x(\alpha+1) + (\alpha^2+1)}{2\pi\alpha\sqrt{(b-x)(x-a)}} \mathbf{1}_{(a,b)}(x)\mathrm{d}x.
$$

Combinatorial analysis of the formula

$$m_k = \sum_{l=1}^{k} \alpha^l |\mathcal{W}_k(k,l,(2,\ldots,2))| + A_4 \sum_{l=1}^{k-1} \alpha^l |\mathcal{W}_k(k-1,l,(4,2,\ldots,2)) + \cdots$$
$$m_k = m_k(\mathrm{MP}_\alpha) + A_4 m_k\left(\mathrm{MP}_\alpha^{\{1\}}\right) + \cdots$$

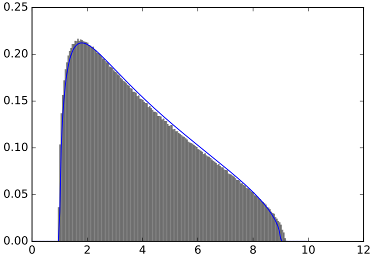
$\mathrm{MP}_\alpha^{\{1\}}$: explicit signed measure with total mass zero.

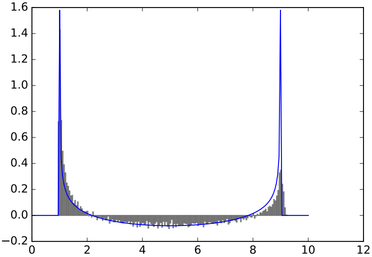
$$\mathrm{MP}_\alpha^{\{1\}}(\mathrm{d}x) = \frac{x^2 - 2x(\alpha+1) + (\alpha^2+1)}{2\pi\alpha\sqrt{(b-x)(x-a)}} \mathbf{1}_{(a,b)}(x)\mathrm{d}x.$$

One example of application: $P_n = \mathrm{Bernoulli}\left(\frac{c}{n}\right)$ where $A_i = c^{1-i/2}$.
Asymptotic development of the limiting law $\mu_{\alpha,c}$ when $c \to +\infty$:

$$m_k(\mu_{\alpha,c}) = m_k(\mathrm{MP}_\alpha) + \frac{1}{c} m_k\left(\mathrm{MP}_\alpha^{\{1\}}\right) + o\left(\frac{1}{c}\right).$$

THANK YOU!





Reference: *Spectra of Wishart's Matrices with Size-Dependent Entries*, N., 2017 (preprint).