The second term for two-neighbour bootstrap percolation in two dimensions

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Les Probabilités de Demain, Paris

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Definition

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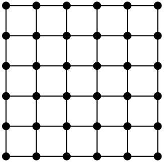
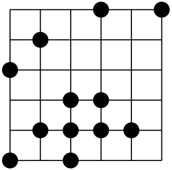
- Graph – $[n]^2 \subset \mathbb{Z}^2$ with n 'large'



Definition

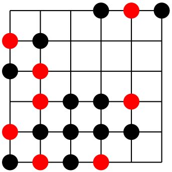
- Graph
- Initial condition – $A_0 \sim \bigotimes_{x\in[n]^2} \mathrm{Bernoulli}(p)$ with p 'small'



Definition

- Graph
- Initial condition
- Bootstrap dynamics – for $t \in \mathbb{N}$

$$A_{t+1} = A_t \cup \{x \in [n]^2, |N_x \cap A_t| \geq 2\}$$



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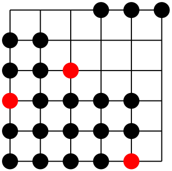

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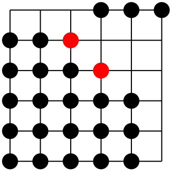

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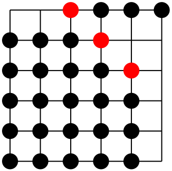

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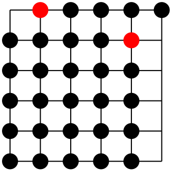

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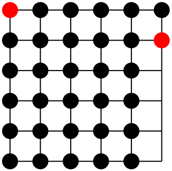

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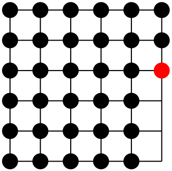

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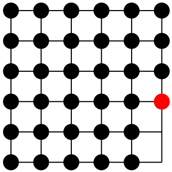

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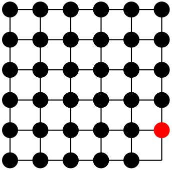

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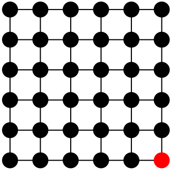
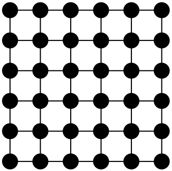


Definition

- Graph
- Initial condition
- Bootstrap dynamics
- Closure – $[A] = \bigcup_{t\in\mathbb{N}} A_t$



Definition

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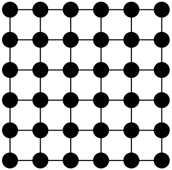
- Initial condition
- Bootstrap dynamics
- Closure
- Percolation event – $[A] = [n]^2$



Definition

- Graph
- Initial condition
- Bootstrap dynamics
- Closure
- Percolation event
- Critical probability –

$$p_c(n) = \inf\{p \in [0,1],\ \mathbb{P}_p(\text{percolation}) \geq 1/2\}$$

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- Zero-temperature dynamics of the Ising model
- Information/disease/... spreading in a network
- Fun

Previous results

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- [Aizenman-Lebowitz'88]

$$\frac{c}{\log n} \leq p_c(n) \leq \frac{C}{\log n}$$

Previous results

- [Aizenman-Lebowitz'88] Scaling

Ideas

- Upper bound: One can infect a square of 'critical' size $1/p$ by finding an infection in each row/column successively. It is found at typical distance $\exp(\Theta(1)/p)$ and easily grows indefinitely.
- Lower bound: Rectangles process – if there is percolation, there exists an internally filled rectangle of every size.

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03]

$$\frac{\pi^2 - \varepsilon}{18 \log n} \leq p_c(n) \leq \frac{\pi^2 + \varepsilon}{18 \log n}$$

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp asymptotics

Ideas

- Upper bound: Only ask for an infection in every second row/column and grow in steps of $1/\sqrt{p}$.
- Lower bound: Hierarchies, disjoint occurrence, pod, quantitative optimality of square shapes ...

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp asymptotics
- [Gravner-Holroyd'08]

$$\frac{\pi^2-\varepsilon}{18\log n}\le p_c(n)\le\frac{\pi^2}{18\log n}-\frac{c}{(\log n)^{3/2}}$$

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp asymptotics
- [Gravner-Holroyd'08] Upper bound for the second term

Idea:
Use the entropy gain from the choice of the lengths of growth steps instead of fixing them as $1/\sqrt{p}$.

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp asymptotics
- [Gravner-Holroyd'08] Upper bound for the second term
- [Gravner-Holroyd-Morris'12]

$$\frac{\pi^2}{18\log n} - \frac{C(\log\log n)^3}{(\log n)^{3/2}} \le p_c(n) \le \frac{\pi^2}{18\log n} - \frac{c}{(\log n)^{3/2}}$$

Previous results

- [Aizenman-Lebowitz'88] Scaling
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- [Gravner-Holroyd'08] Upper bound for the second term
- [Gravner-Holroyd-Morris'12] Almost matching lower bound

Idea:
Consider finer hierarchies starting from size $1/\sqrt{p}$. Compensate the large number of hierarchies with the high cost of having many large seeds.

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp asymptotics
- [Gravner-Holroyd'08] Upper bound for the second term
- [Gravner-Holroyd-Morris'12] Almost matching lower bound
- [Bringmann-Mahlburg'12] 'morally'

$$\frac{\pi^2}{18\log n} - \frac{C(\log\log n)^{5/2}}{(\log n)^{3/2}} \leq p_c(n) \leq \frac{\pi^2}{18\log n} - \frac{c}{(\log n)^{3/2}}$$

Previous results

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- [Bringmann-Mahlburg'12]
- [Balogh-Bollobás'03] Critical window size is

$$\frac{(\log \log n)^{O(1)}}{(\log n)^2}$$

Previous results

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- [Bringmann-Mahlburg'12]
- [Balogh-Bollobás'03] Window size

Idea:
Apply [Friedgut-Kalai'96].

Previous results

- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp asymptotics
- [Gravner-Holroyd'08] Upper bound for the second term
  'Morally' the critical window is

$$\frac{\Theta(1)}{(\log n)^2}$$

- [Gravner-Holroyd-Morris'12] Almost matching lower bound
- [Bringmann-Mahlburg'12]
- [Balogh-Bollobás'03] Window size

Theorem (H, Morris'19)

$$p_c(n) = \frac{\pi^2}{18 \log n} - \frac{\Theta(1)}{(\log n)^{3/2}}$$

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Remark

The upper bound is from GH.

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- Key lemmas – strong bounds on the probability of gradual growth.
- Multiple pods allow taking advantage of atypical rectangles featuring in hierarchies.
- Optimised amount of growth of a rectangle in one step depending on its size. In particular, a swift divergence is needed above the critical size.

- Bounded number of (large) seeds.
- Small pod.
- Short hierarchies.
- Non-small rectangles are not far from squares.

- What is the constant?
- Is the next error term the window size?
- Can similar results be obtained for higher thresholds (in higher dimensions)?
- Can similar results be obtained for other (critical balanced) models?

Thank you.

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