Stein's method and Malliavin calculus for independent random variables

Hélène Halconruy

under the supervision of Laurent Decreusefond

Télécom ParisTech - Université de Paris Saclay

TELECOM ParisTech

4e édition LES PROBABILITÉS DE DEMAIN
SYLVIE MÉLÉARD | STÉPHANE SEURET

Introduction

Theorem (Central Limit Theorem)

Let $(X_n,\, n \geq 1)$ *be a sequence of i.i.d. twice integrable random variables defined on* $(\Omega, \mathcal{A}, \mathbf{P})$

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) - \mathbf{E}\left[X_1\right]\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \operatorname{var}[X_1]).$$

Introduction

Theorem (Central Limit Theorem)

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. twice integrable random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$

$$\sqrt{n}\left(\Big(\frac{1}{n}\sum_{k=1}^{n} X_k\Big) - \mathbf{E}\left[X_1\right]\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \mathrm{var}[X_1]).$$

→ Rate of convergence for the law of numbers.

Introduction

Theorem (Central Limit Theorem)

Let $(X_n,\, n \geq 1)$ *be a sequence of i.i.d. twice integrable random variables defined on* $(\Omega, \mathcal{A}, \mathbf{P})$

$$\underbrace{\sqrt{n}\left(\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) - \mathbf{E}\left[X_1\right]\right)}_{F(X_1,\cdots,X_n)} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \mathrm{var}[X_1]).$$

→ Rate of convergence for the law of numbers.

Question : how to estimate $\mathrm{dist}(F^*\mathbf{P}_n, \mathbf{P})$ where $\mathbf{P}_n := \otimes_{k=1}^{n} \mathbf{P}_{X_k}$ and for $\mathbf{P}, \mathbf{Q}$ measures on a Polish space $\mathfrak{F}$

$$\mathrm{dist}_W(\mathbf{P}, \mathbf{Q}) = \sup_{g \in \mathcal{T}} \left| \int g \, \mathrm{d}\mathbf{P} - \int g \, \mathrm{d}\mathbf{Q} \right|$$

and $\mathcal{T} = \{g \in \mathcal{C}^1(\mathbf{R}, \mathbf{R}) \,:\, \|g'\|_\infty \leq 1\}$?

Introduction

Theorem (Central Limit Theorem)

Let $(X_n,\, n \geq 1)$ *be a sequence of i.i.d. twice integrable random variables defined on* $(\Omega, \mathcal{A}, \mathbf{P})$

$$\underbrace{\sqrt{n}\left(\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) - \mathbf{E}\left[X_1\right]\right)}_{F(X_1,\cdots,X_n)} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \mathrm{var}[X_1]).$$

→ Rate of convergence for the law of numbers.

Question : how to estimate $\mathrm{dist}(F^*\mathbf{P}_n, \mathbf{P})$ where $\mathbf{P}_n := \otimes_{k=1}^{n} \mathbf{P}_{X_k}$ and for $\mathbf{P}, \mathbf{Q}$ measures on a Polish space $\mathfrak{F}$

$$\mathrm{dist}_W(\mathbf{P}, \mathbf{Q}) = \sup_{g\in\mathcal{T}} \left| \int g \,\mathrm{d}\mathbf{P} - \int g \,\mathrm{d}\mathbf{Q} \right|$$

and $\mathcal{T} = \{g \in \mathcal{C}^1(\mathbf{R}, \mathbf{R}) \,:\, \|g'\|_\infty \leq 1\}$?

Stein's method (1)

- Caracterization of the target measure $\mathbf{P}$.

$$\text{For all } g \in \mathcal{T},\ \int \mathcal{L}g \,\mathrm{d}\mathbf{Q} = 0 \iff \mathbf{Q} = \mathbf{P}.$$

  $\mathcal{L}$ is the *Stein operator* associated to $\mathbf{P}$.

Stein's method (1)

- Caracterization of the target measure $\mathbf{P}$.
  $$\text{For all } g \in \mathcal{T}, \int \mathcal{L}g \, \mathrm{d}\mathbf{Q} = 0 \iff \mathbf{Q} = \mathbf{P}.$$
  $\mathcal{L}$ is the *Stein operator* associated to $\mathbf{P}$.
- Resolution of the Stein equation
  $$\mathcal{L}\varphi_g = g - \int g \, \mathrm{d}\mathbf{P}, \quad (1)$$
  for any $g \in \mathcal{T}$, then
  $$g \in \mathcal{T} \iff \varphi_g \in \mathcal{F}.$$

Stein's method (1)

- Caracterization of the target measure $\mathbf{P}$.

  For all $g \in \mathcal{T}$, $\displaystyle\int \mathcal{L}g \,\mathrm{d}\mathbf{Q} = 0 \iff \mathbf{Q} = \mathbf{P}$.

  $\mathcal{L}$ is the *Stein operator* associated to $\mathbf{P}$.

- Resolution of the Stein equation

$$\mathcal{L}\varphi_g = g - \int g \,\mathrm{d}\mathbf{P}, \tag{1}$$

  for any $g \in \mathcal{T}$, then

$$g \in \mathcal{T} \iff \varphi_g \in \mathcal{F}.$$

- Equivalent problem

$$\sup_{g\in\mathcal{T}} \left| \int g \,\mathrm{d}\mathbf{Q} - \int g \,\mathrm{d}\mathbf{P} \right| = \sup_{\varphi\in\mathcal{F}} \left| \mathbf{E}\left[\mathcal{L}\varphi(X)\right] \right| \quad \text{where } X \sim \mathbf{Q}.$$

Stein's method (1)

- Caracterization of the target measure $\mathbf{P}$.

  For all $g \in \mathcal{T}$, $\int \mathcal{L}g \, \mathrm{d}\mathbf{Q} = 0 \iff \mathbf{Q} = \mathbf{P}$.

  $\mathcal{L}$ is the *Stein operator* associated to $\mathbf{P}$.

- Resolution of the Stein equation

$$\mathcal{L}\varphi_g = g - \int g \, \mathrm{d}\mathbf{P}, \tag{1}$$

  for any $g \in \mathcal{T}$, then

$$g \in \mathcal{T} \iff \varphi_g \in \mathcal{F}.$$

- Equivalent problem

$$\sup_{g \in \mathcal{T}} \left| \int g \, \mathrm{d}\mathbf{Q} - \int g \, \mathrm{d}\mathbf{P} \right| = \sup_{\varphi \in \mathcal{F}} \left| \mathbf{E}\left[\mathcal{L}\varphi(X)\right] \right| \quad \text{where } X \sim \mathbf{Q}.$$

Stein's method (2)

- Principle :

$$\mathrm{dist}_{\mathcal{T}}(\mathbf{P}, \mathbf{Q}) = \sup_{\varphi \in \mathcal{F}} |\mathbf{E}[\mathcal{L}\varphi(X)]|$$

Stein's method (2)

- Principle :

$$\begin{aligned}
\mathrm{dist}_{\mathcal{T}}(\mathbf{P}, \mathbf{Q}) &= \sup_{\varphi \in \mathcal{F}} |\mathbf{E}\left[\mathcal{L}\varphi(X)\right]| \\
&= \sup_{\varphi \in \mathcal{F}} |\mathbf{E}\left[\mathcal{L}_1\varphi(X)\right] + \mathbf{E}\left[\mathcal{L}_2\varphi(X)\right]|
\end{aligned}$$

Stein's method (2)

- Principle :

$$\text{dist}_{\mathcal{T}}(\mathbf{P}, \mathbf{Q}) = \sup_{\varphi \in \mathcal{F}} |\mathbf{E}[\mathcal{L}\varphi(X)]| = \sup_{\varphi \in \mathcal{F}} |\mathbf{E}[\mathcal{L}_1\varphi(X)] + \mathbf{E}[\mathcal{L}_2\varphi(X)]|$$

- Idea : transform $\mathcal{L}_1\varphi(X)$ into $-\mathcal{L}_2\varphi(X)+$ remainder
  $\rightarrow$ rate of convergence.
  3 methods :
  1. Exchangeable pairs.
  2. Size-biased.
  3. Malliavin integration by parts.

Stein's method (2)

- Principle :

$$\begin{aligned}\mathrm{dist}_{\mathcal{T}}(\mathbf{P},\mathbf{Q}) &= \sup_{\varphi\in\mathcal{F}} \left|\mathbf{E}\left[\mathcal{L}\varphi(X)\right]\right| \\ &= \sup_{\varphi\in\mathcal{F}} \left|\mathbf{E}\left[\mathcal{L}_1\varphi(X)\right] + \mathbf{E}\left[\mathcal{L}_2\varphi(X)\right]\right|\end{aligned}$$

- Idea : transform $\mathcal{L}_1\varphi(X)$ into $-\mathcal{L}_2\varphi(X)+$ remainder
  $\rightarrow$ rate of convergence.
  3 methods :
  1. Exchangeable pairs.
  2. Size-biased.
  3. Malliavin integration by parts.

Finite dimension differential calculus and Malliavin calculus
Analogous terminology

Differential calculus on Classical Euclidian spaces	**Malliavin calculus on Wiener space**
Vectors	
Functions	
Gradient	

Finite dimension differential calculus and Malliavin calculus
Analogous terminology

Differential calculus on Classical Euclidian spaces	**Malliavin calculus on Wiener space**
Vectors	Paths of Brownian motion
Functions	
Gradient	

Finite dimension differential calculus and Malliavin calculus
Analogous terminology

Differential calculus on Classical Euclidian spaces	**Malliavin calculus on Wiener space**
Vectors	Paths of Brownian motion
Functions	Random variables = Functionals of the paths
Gradient	

Finite dimension differential calculus and Malliavin calculus
Analogous terminology

Differential calculus on Classical Euclidian spaces	**Malliavin calculus on Wiener space**
Vectors	Paths of Brownian motion
Functions	Random variables = Functionals of the paths
Gradient	Malliavin derivative

Stein-Malliavin criterion on the Gaussian space

- $H = \mathrm{L}^2(T, \mathcal{B}, \mu)$ real separable Hilbert space.
- $X = \{X(h),\, h \in H\}$ centered I.G.P. and $\tilde{\mathbf{Q}} = X^*\mathbf{P}$.
- $\mathcal{S}$: space of cylindrical random variables of the form

$$F = f(X(h_1), \ldots, X(h_n))\,;\; f \in \mathcal{C}_c(\mathbf{R}^n, \mathbf{R}),\; h_i \in H.$$

For $F \in \mathcal{S}$, $DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), \ldots, X(h_n))\, h_i$.

- δ adjoint of D and $L = -\delta D$ "Laplacian operator".

Theorem (Nourdin, Peccati)

For any $F \in \mathbf{D}^{1,2}$ *with* $\mathbf{E}[F] = 0$,

$$\mathrm{dist}_W\big(F^*\tilde{\mathbf{Q}},\, \mathbf{P}\big) \le \left(\mathbf{E}\left[\left|1 - \langle DF\,,\, -DL^{-1}F\rangle_H\right|^2\right]\right)^{\frac{1}{2}}.$$

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \leq 1,\, \|\varphi''\|_\infty \leq 2\}$.

$$\mathbf{E}\left[\overbrace{\underbrace{F}_{L(L^{-1}F)} \quad \varphi(F)}^{\mathcal{L}_1\varphi(F)}\right]$$

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\, \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\left[\overbrace{\underbrace{F}_{L(L^{-1}F)}\quad\varphi(F)}^{\mathcal{L}_1\varphi(F)}\right] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right]$$

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\, \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\left[\overbrace{\underbrace{F}_{L(L^{-1}F)}\quad \varphi(F)}^{\mathcal{L}_1\varphi(F)}\right] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right] \overset{\text{I.P.P.}}{=} \mathbf{E}\left[\langle D\varphi(F)\,,\, -DL^{-1}F\rangle_H\right]$$

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\ \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\Bigg[\overbrace{\underbrace{F}_{L(L^{-1}F)}\ \varphi(F)}^{\mathcal{L}_1\varphi(F)}\Bigg] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right] \overset{\text{I.P.P.}}{=} \mathbf{E}\left[\langle D\varphi(F)\,,\,-DL^{-1}F\rangle_H\right]$$

$$= \mathbf{E}\left[\varphi'(F)\langle DF\,,\,-DL^{-1}F\rangle_H\right] \quad \text{(chain rule)}$$

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\ \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\Big[\overbrace{\underbrace{F}_{L(L^{-1}F)}\ \varphi(F)}^{\mathcal{L}_1\varphi(F)}\Big] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right] \overset{\text{I.P.P.}}{=} \mathbf{E}\left[\langle D\varphi(F)\,,\,-DL^{-1}F\rangle_H\right]$$

$$= \mathbf{E}\left[\varphi'(F)\langle DF\,,\,-DL^{-1}F\rangle_H\right] \quad \text{(chain rule)}$$

$$= \mathbf{E}\Big[\ \overbrace{\varphi'(F)}^{-\mathcal{L}_2\varphi(F)}\ \overbrace{-\varphi'(F)\big(1+\langle DF\,,\,-DL^{-1}F\rangle_H\big)}^{\text{remainder}}\Big]$$

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\, \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\Bigg[\overbrace{\underbrace{F}_{L(L^{-1}F)}\ \varphi(F)}^{\mathcal{L}_1\varphi(F)}\Bigg] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right] \overset{\text{I.P.P.}}{=} \mathbf{E}\left[\langle D\varphi(F)\,,\,-DL^{-1}F\rangle_H\right]$$

$$= \mathbf{E}\left[\varphi'(F)\langle DF\,,\,-DL^{-1}F\rangle_H\right] \quad \text{(chain rule)}$$

$$= \mathbf{E}\Big[\ \overbrace{\varphi'(F)}^{-\mathcal{L}_2\varphi(F)}\ \overbrace{-\varphi'(F)\big(1 + \langle DF\,,\,-DL^{-1}F\rangle_H\big)}^{\text{remainder}}\Big]$$

Tools

- A caracterization of $\mathbf{P}$ in terms of 1^{st}-order differential operators.

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\, \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\left[\overbrace{\underbrace{F}_{L(L^{-1}F)}\ \varphi(F)}^{\mathcal{L}_1\varphi(F)}\right] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right] \overset{\text{I.P.P.}}{=} \mathbf{E}\left[\langle D\varphi(F)\,,\,-DL^{-1}F\rangle_H\right]$$

$$= \mathbf{E}\left[\varphi'(F)\langle DF\,,\,-DL^{-1}F\rangle_H\right] \quad \text{(chain rule)}$$

$$= \mathbf{E}\Big[\overbrace{\varphi'(F)}^{-\mathcal{L}_2\varphi(F)}\ \overbrace{-\varphi'(F)\big(1+\langle DF\,,\,-DL^{-1}F\rangle_H\big)}^{\text{remainder}}\Big]$$

<u>**Tools**</u>

- A caracterization of $\mathbf{P}$ in terms of 1^{st}-order differential operators.
- A Malliavin derivative operator D.

Sketch of the proof

- $\mathcal{L}\varphi(x) = x\varphi(x) - \varphi'(x) =: \mathcal{L}_1\varphi(x) + \mathcal{L}_2\varphi(x)$.
- $\mathcal{F} = \{\varphi \in \mathcal{C}^2(\mathbf{R},\mathbf{R}) \,:\, \|\varphi'\|_\infty \le 1,\, \|\varphi''\|_\infty \le 2\}$.

$$\mathbf{E}\left[\overbrace{\underbrace{F}_{L(L^{-1}F)}\;\varphi(F)}^{\mathcal{L}_1\varphi(F)}\right] = \mathbf{E}\left[-\delta(DL^{-1}F)\,\varphi(F)\right] \overset{\text{I.P.P.}}{=} \mathbf{E}\left[\langle D\varphi(F)\,,\,-DL^{-1}F\rangle_H\right]$$

$$= \mathbf{E}\left[\varphi'(F)\langle DF\,,\,-DL^{-1}F\rangle_H\right] \quad \text{(chain rule)}$$

$$= \mathbf{E}\Big[\overbrace{\varphi'(F)}^{-\mathcal{L}_2\varphi(F)}\;\overbrace{-\varphi'(F)\big(1+\langle DF\,,\,-DL^{-1}F\rangle_H\big)}^{\text{remainder}}\Big]$$

Tools

- A caracterization of $\mathbf{P}$ in terms of 1^{st}-order differential operators.
- A Malliavin derivative operator D.
- An integration by parts formula including δ.

Stein-Malliavin criterion on a discrete setting

Question : how to estimate

$$\mathrm{dist}_W\left(\tilde{\mathbf{Q}}, \mathbf{P}\right)$$

if $\tilde{\mathbf{Q}} = F^*\mathbf{P}_A$ where $\mathbf{P}_A = \otimes_{a\in A}\mathbf{P}_a$ (A countable set) and F is a functional of independent random variables ?

Stein-Malliavin criterion on a discrete setting

Question : how to estimate

$$\mathrm{dist}_W\left(\tilde{\mathbf{Q}}, \mathbf{P}\right)$$

if $\tilde{\mathbf{Q}} = F^*\mathbf{P}_A$ where $\mathbf{P}_A = \otimes_{a\in A}\mathbf{P}_a$ (A countable set) and F is a functional of independent random variables ?

... And then to construct a Malliavin calculus on a denumerable product of probability space $(\mathrm{E}_A, \mathcal{E}_A, \mathbf{P}_A)$ where $\mathrm{E}_A = \prod_{a\in A}\mathrm{E}_a, \mathcal{E}_A = \bigvee_{a\in A}\mathcal{E}_a$ and $\mathbf{P}_A = \otimes_{a\in A}\mathbf{P}_a$?

Gradient

$\mathcal{S}$: space of cylindrical random variables of the form

$$F = F(X_1, \dots, X_n).$$

Definition

For $F \in \mathcal{S}$, $DF \in L^2(A \times \mathrm{E}_A)$ *is defined by : For all* $a \in A$,

$$\begin{aligned} D_a F(X_A) &= F(X_A) - \mathbf{E}\left[F(X_A) \,|\, \mathcal{G}_a\right] \\ &= F(X_A) - \mathbf{E}'\left[F(X_{A \setminus a}, X'_a)\right] \end{aligned}$$

where $\mathcal{G}_a = \sigma\{X_b,\, b \neq a\}$ *and* X'_a *is an independent copy of* X_a.

(2)

Gradient

$\mathcal{S}$: space of cylindrical random variables of the form

$$F = F(X_1, \ldots, X_n).$$

Definition

For $F \in \mathcal{S}$, $DF \in L^2(A \times \mathrm{E}_A)$ *is defined by : For all* $a \in A$,

$$\begin{aligned} D_a F(X_A) &= F(X_A) - \mathbf{E}\left[F(X_A)\,|\,\mathcal{G}_a\right] \\ &= F(X_A) - \mathbf{E}'\left[F(X_{A\smallsetminus a}, X'_a)\right] \end{aligned}$$

where $\mathcal{G}_a = \sigma\{X_b,\, b \neq a\}$ *and* X'_a *is an independent copy of* X_a.

- The domain of D, $\mathbf{D}$ is the closure of $\mathcal{S}$ w.r.t. the norm

$$\|F\|_{\mathbf{D}}^2 = \|F\|^2_{\mathrm{L}^2(\mathrm{E}_A)} + \|DF\|^2_{\mathrm{L}^2(A\times\mathrm{E}_A)}$$

- Property : for $F, G \in \mathcal{S}$,

$$\mathbf{E}\left[\sum_{a\in A} D_a F \cdot G\right] = \mathbf{E}\left[F \sum_{a\in A} D_a G\right] \tag{2}$$

Gradient

$\mathcal{S}$: space of cylindrical random variables of the form

$$F = F(X_1, \ldots, X_n).$$

Definition

For $F \in \mathcal{S}$, $DF \in L^2(A \times \mathrm{E}_A)$ is defined by : For all $a \in A$,

$$\begin{aligned} D_a F(X_A) &= F(X_A) - \mathbf{E}\left[F(X_A) \,|\, \mathcal{G}_a\right] \\ &= F(X_A) - \mathbf{E}'\left[F(X_{A \setminus a}, X'_a)\right] \end{aligned}$$

where $\mathcal{G}_a = \sigma\{X_b,\, b \neq a\}$ and X'_a is an independent copy of X_a.

- The domain of D, $\mathbf{D}$ is the closure of $\mathcal{S}$ w.r.t. the norm

$$\|F\|_{\mathbf{D}}^2 = \|F\|_{\mathrm{L}^2(\mathrm{E}_A)}^2 + \|DF\|_{\mathrm{L}^2(A \times \mathrm{E}_A)}^2$$

- Property : for $F, G \in \mathcal{S}$,

$$\mathbf{E}\left[\sum_{a \in A} D_a F \cdot G\right] = \mathbf{E}\left[F \sum_{a \in A} D_a G\right] \qquad (2)$$

Divergence and integration by parts formula

Definition (Divergence)

For any $U \in \operatorname{Dom}\delta$, $\delta U \in \mathrm{L}^2(\mathrm{E}_A)$

$$\langle DF, U\rangle_{\mathrm{L}^2(A\times \mathrm{E}_A)} = \langle F, \delta U\rangle_{\mathrm{L}^2(\mathrm{E}_A)}, \text{ for all } F \in \mathbf{D}. \tag{3}$$

By property (2),

$$\delta U = \sum_{a\in A} D_a U_a.$$

Theorem (Integration by parts formula - LD, HH)

For any $F \in \mathbf{D}$, $U \in \operatorname{Dom}\delta$,

$$\mathbf{E}\left[\sum_{a\in A} D_a F\cdot U_a\right] = \mathbf{E}\left[F\cdot \delta U\right].$$

Normal Approximation

- $\mathbf{P} = \mathcal{N}(0,1)$.
- $\mathcal{T} = \{h \in \mathcal{C}^1(\mathbf{R},\mathbf{R}) \,:\, \|h'\|_\infty \le 1\}$.

Theorem (LD, HH)

For any $F : \mathrm{E}_A \to \mathbf{R}$, s.t. $F \in \mathbf{D}$ and $\mathbf{E}[F] = 0$,

$$\operatorname{dist}_W\left(F^*\mathbf{P}_A,\, \mathbf{P}\right) \le \mathbf{E}\left[\left|1 - \sum_{a\in A} D_aF\ (-D_aL^{-1})F\right|\right] + \sum_{a\in A}\mathbf{E}\left[\int_{\mathrm{E}_A}\Big(F - F(X_{A\neg a};x)\Big)^2 \mathrm{d}\mathbf{P}_a(x)\ |D_aL^{-1}F|\right].$$

where $X'_{\neg a} = X_{A\neg a} \cup \{X'_a\}$ and $L = -\delta D$ is the number operator.

Normal Approximation

- $\mathbf{P} = \mathcal{N}(0, 1)$.
- $\mathcal{T} = \{h \in \mathcal{C}^1(\mathbf{R}, \mathbf{R}) \, : \, \|h'\|_\infty \leq 1\}$.

Theorem (LD, HH)

For any $F : \mathrm{E}_A \to \mathbf{R}$, s.t. $F \in \mathbf{D}$ and $\mathbf{E}[F] = 0$,

$$\mathrm{dist}_W\big(F^*\mathbf{P}_A,\, \mathbf{P}\big) \leq \mathbf{E}\left[\left|1 - \sum_{a \in A} D_a F \; (-D_a L^{-1})F\right|\right] + \sum_{a \in A} \mathbf{E}\left[\int_{\mathrm{E}_A} \Big(F - F(X_{A\neg a}; x)\Big)^2 \mathrm{d}\mathbf{P}_a(x) \; |D_a L^{-1} F|\right].$$

where $X'_{\neg a} = X_{A\neg a} \cup \{X'_a\}$ and $L = -\delta\, D$ is the number operator.

Application : Lyapounov theorem

Corollary (Lyapounov theorem - LD, HH)

Let $(X_n,\, n \geq 1)$ be a sequence of thrice integrable independent random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$. Denote

$$\sigma_n^2 = \operatorname{var}(X_n),\; s_n^2 = \sum_{j=1}^{n} \sigma_j^2 \;\text{ and }\; Y_n = \frac{1}{s_n} \sum_{j=1}^{n} (X_j - \mathbf{E}[X_j])\,.$$

Assume that

$$\lim_{n\to\infty} \frac{1}{s_n^3} \sum_{j=1}^{n} \mathbf{E}\left[|X_j - \mathbf{E}[X_j]|^3\right] = 0. \tag{4}$$

Then,

$$\operatorname{dist}_W(\mathbf{P}_{Y_n}, \mathbf{P}) \leq \frac{2(\sqrt{2}+1)}{s_n^3} \sum_{j=1}^{n} \mathbf{E}\left[|X_j - \mathbf{E}[X_j]|^3\right].$$

References

- L. Decreusefond, *The Stein-Dirichlet-Malliavin method.* ESAIM: Proceedings, 2015.
- I. Nourdin, G. Peccati, *Stein's method on Wiener chaos*, Probab. Theory Related Fields.
- G. Peccati, J.L. Solé, M.S. Taqqu, F. Utzet, *Stein's method and normal approximation of Poisson functionals.* Annals of Probability, 2010.
- N. Privault, *Stochastic analysis in discrete and continuous settings with normal martingales.* Springer-Verlag, Berlin, 2009.
- L. Decreusefond, H. Halconruy, *Malliavin and Dirichlet structures for independent random variables.* Stochastic Processes and their Applications, 2019.

Thanks for your attention !