Strong solutions of semilinear SPDEs with unbounded diffusion[1]

Florian Bechtold

LPSM – Sorbonne Université

Les probabilités de demain
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SPDEs as infinite dimensional SDEs

For $a, b \in \mathbb{R}$ the stochastic differential equation

$$\begin{cases} dX_t &= aX_t dt + bX_t dW_t \\ X_0 &= x \in \mathbb{R} \end{cases}$$

admits a unique solution (geometric Brownian motion), which satisfies moreover the equation

$$X_t = e^{at}x + \int_0^t e^{a(t-s)} bX_s dW_s$$

To consider the problem in infinite dimensions, do the replacements

Real number a	Linear (unbounded) operator A
Exponential of real number a	Semigroup $(S_t)_t$ generated by A
Real number b	Nonlinear bounded operator B
Brownian motion	Cylindrical Brownian motion

Cylindrical Brownian motion and stochastic integration in infinite dimensions

Formally, for a Hilbert space U with ONB $(e_k)_k$, set for a sequence $(\beta^k)_k$ of independet Brownian motions as cylindrical Brownian motion

$$W_t := \text{"} \sum_{k=1}^{\infty} e_k \beta_t^k \text{"} \in U$$

Problem: This is not an element of $L^2(\Omega, U)$.
Suppose for $B \in L(U, H)$ and for an ONB $(f_k)_k$ of H, there exists $(\lambda_k)_k \subset \mathbb{R}$ such that

$$Bu = \sum_{k}^{\infty} \lambda_k f_k \langle u, e_k \rangle$$

Now if $(\lambda_k)_k \in \ell^2$, then $B \in L_2(U, H)$ and

$$\mathbb{E}[|| \int_0^t B dW_s ||_H^2] = \mathbb{E}[|| \sum_k \lambda_k f_k \beta_t^k ||_H^2] = \sum_k \lambda_k^2 t < \infty$$

The mild formulation for semilinear SPDEs

Definition (Mild solution)

Let H be Hilbert, $A : D(A) \subset H \to H$ be generator of a strongly continuous semigroup $(S_t)_{t \leq T}$ of operators and $B : H \to L_2(U, H)$, let W be a cylindrical Wiener process on U. Then a progressively measurable process $(u_t)_{t \leq T}$ with values in H is called mild solution to

$$du_t = Au_t dt + B(u_t) dW_t, \qquad u_0 = \xi \in H$$

if $\mathbb{P}$-almost surely for all $t \leq T$

$$u_t = \underbrace{S_t \xi + \int_0^t S_{t-s} B(u_s) dW_s}_{=: (\mathcal{K}(u))_t}$$

holds as an equality in H.

The key tool: A maximal inequality

Lemma (Maximal inequality for stochastic convolutions)

Let H be Hilbert, $u \in L^p(\Omega, L^\infty([0,T],H))$, let $B : H \to L_2(U,H)$, let W be a U-cylindrical Brownian motion, then for all contraction semi groups S on H one has

$$\mathbb{E}[\sup_{t\leq T} || \int_0^t S_{t-s}B(u_s)dW_s||_H^p] \leq CT^{p/2}\mathbb{E}[\sup_{t\leq T} ||B(u_s)||^p_{L_2(U,H)}].$$

Corollary (Existence and uniqueness of mild solutions)

If $B : H \to L_2(U,H)$ is Lipschitz and of linear growth, then

$$\mathcal{K} : L^p(\Omega, L^\infty([0,T],H)) \to L^p(\Omega, L^\infty([0,T],H))$$

admits a unique fixed point for T sufficiently small, i.e. the corresponding SPDE admits a unique mild solution.

Our problem

It is natural to ask, if one can also consider unbounded diffusions, i.e. problems of the form

$$\begin{cases} du & = \Delta u dt + (-\Delta)^{\delta/2} B(u_t) dW_t \\ u(0) & = u_0 \in L^p(\mathbb{T}^N) \end{cases}$$

for $\delta \in [0, 1)$. The corresponding mild formulation becomes

$$u_t = S_t u_0 + \int_0^t (-\Delta)^{\delta/2} S_{t-s} B(u_s) dW_t$$

Can one replace the maximal inequality for stochastic convolutions with something more general, adapted to this context?

A maximal inequality and mild solutions for $\delta \in [0,1)$

Theorem (A maximal inequality)

Let H be Hilbert. Let $A : D(A) \subset H \to H$ be generator of a strongly continuous contraction semi-group of operators $(S_t)_{t \leq T}$. Let $B : H \to L_2(U,H)$ satisfy

$$||B(z)||^2_{L_2(U,H)} \leq C(1 + ||z||^2_H).$$

Let W be a cylindrical Wiener process on U. Let $\delta \in [0,1)$, $T > 0$. Then for $p > \frac{2}{1-\delta}$ and every progressively measurable $u \in L^p(\Omega, L^\infty([0,T], H))$ there holds

$$\mathbb{E}[\sup_{t \leq T} || \int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s ||^p_H] \leq C_{\delta,p} T^{\frac{p}{2}(1-\delta)} \mathbb{E}[\sup_{t \leq T} ||B(u_t)||^p_{L_2(U,H)}]$$

and the associated convolution process is continuous.

A maximal inequality and mild solutions for $\delta \in [0, 1)$

Theorem (General existence theorem)

Let H be Hilbert. Let $B : H \to L_2(U, H)$ be Lipschitz continuous, satisfying the growth condition previously stated. Then for $\delta \in [0, 1)$ and $p > \frac{2}{1-\delta}$ the SPDE

$$du_t = \Delta u_t dt + (-\Delta)^{\delta/2} B(u_t) dW_t \qquad u(0) = u_0 \in H$$

admits a unique mild solution, meaning there exists a unique progressively measurable process $u \in L^p(\Omega, L^\infty([0, T], H))$ such that

$$u_t = \underbrace{S_t u_0 + \int_0^t (-\Delta)^{\delta/2} S_{t-s} B(u_s) dW_s}_{=:(\mathcal{K}(u))_t}$$

which is $\mathbb{P}$-almost surely continuous in time.

A maximal inequality and mild solutions for $\delta \in [0,1)$

Idea of the proof:
Use Banach fixed-point theorem in the space
$Z_H := L^p(\Omega, L^\infty([0,T], H))$ exploiting the maximal inequality

$$\begin{aligned}
||\mathcal{K}(u) - \mathcal{K}(v)||^p_{Z_H} &= \mathbb{E}[\sup_{s\le T} || \int_0^s (-\Delta)^{\delta/2} S_{s-r} B(u_r) - B(v_r) dW_r||^p_H] \\
&\le C_{\delta,p} T^{p/2(1-\delta)} \mathbb{E}[\sup_{s\le t} ||B(u_s) - B(v_s)||^p_{L_2(U,H)}] \\
&\le C_{\delta,p} T^{p/2(1-\delta)} \mathbb{E}[\sup_{s\le t} ||u_s - v_s||^p_H] \\
&= C_{\delta,p} T^{p/2(1-\delta)} ||u - v||^p_{Z_H}
\end{aligned}$$

For $\delta < 1$, one can choose T sufficiently small for $\mathcal{K}$ to become a contraction.

Nemytskii operators and strong solutions for $\delta \in [0, 1)$

While the previous theorem works for infinite dimensional noise, we restrict ourselves in the following to finite dimension noise, i.e. U is of finite dimension, for some basis $(e_i)_{e \leq d}$ we have

$$B(u_s)dW_s = \sum_{i=1}^{d} B_i(u_s)dW_s^i \left(= \sum_{i=1}^{d} B_i(u_s)\langle dW_s, e_i \rangle \right)$$

The question thus becomes: under which conditions on the functions $(B_i)_{i \leq d}$ does the associated Nemytskii operator have the required properties, more precisely:
Given the Hilbert space $L^2(\mathbb{T}^N)$, under what conditions on a real valued function $B_i : \mathbb{R} \to \mathbb{R}$ is the operator

$$B : L^2(\mathbb{T}^N) \to L_2(U, L^2(\mathbb{T}^N))$$

$$z \to \left(u \to \sum_{i=1}^{d} B_i(z)\langle u, e_i \rangle \right)$$

Lipschitz and satisfying the growth condition?

Nemytskii operators and strong solutions for $\delta \in [0, 1)$

Lemma (A Nemytskii operator type result for $L^2(\mathbb{T}^N)$)

Let U be a d-dimensional Hilbert space with ONB $(e_i)_{i \leq d}$. Let $B_1, \dots, B_d \in C^1(\mathbb{R}; \mathbb{R})$ be of bounded derivative and satisfy the growth condition

$$\sum_{i=1}^{d} |B_i(\xi)|^2 \leq C(1 + |\xi|^2)$$

Then the associated Nemytskii operator is well defined, Lipschitz and satisfies the growth condition

$$||B(z)||^2_{L_2(U, L^2(\mathbb{T}^N))} \leq C(1 + ||z||^2_{L^2(\mathbb{T}^N)}).$$

Nemytskii operators and strong solutions for $\delta \in [0, 1)$

Problem in passing from Lebesgue to Sobolev spaces:
Nemytskii operators lose Lipschitz continuity.

Yet, the growth condition can be preserved, which is sufficient

- Picard iterations uniformly bounded (geometric series for T small)
- Extraction of a weak-*-convergent subsequence (in Sobolev space)
- Argue via uniqueness of limit

Thanks for your attention!

Starting point

In her first publication, Hofmanova considered a class of problems comprising in particular

$$\begin{cases} du &= \Delta u dt + \text{div} F(u) dt + \sum_{k=1}^{d} B_k(u) dW_t^k \\ u(0) &= u_0 \end{cases}$$

By using the mild formulation, she can

- Control the divergence by the smoothing of the semigroup
- Establish regularity in space of the unique solution thanks to the Sobolev embedding theorem (requires the theory of stochastic integration in 2-smooth Banach spaces)

Hofmanova's setting

The mild formulation corresponding to Hofmanova's problem

$$du = \Delta u_t dt + \text{div} F(u_t) dt + B(u_t) dW_t \qquad u_0 = \xi$$

is given by

$$\begin{aligned} u_t &= S_t \xi + \int_0^t S_{t-s} \text{div} F(u_s) ds + \int_0^t S_{t-s} B(u_s) dW_s \\ &= S_t \xi + \int_0^t (-\Delta)^{1/2} S_{t-s} (-\Delta)^{-1/2} \text{div} F(u_s) ds + \int_0^t S_{t-s} B(u_s) dW_s \end{aligned}$$

Key idea in Hofmanova's paper to deal with Bochner integral:

- $||(-\Delta)^{1/2} S_{t-s}|| \leq C \frac{1}{\sqrt{t-s}}$
- $(-\Delta)^{-1/2} \text{div} F(\cdot)$ is a bounded operator
- Use trinangle inequality, get control via time horizon T ($\rightarrow$ think contraction for fixed point argument)

A maximal inequality and mild solutions for $\delta \in [0,1)$

Idea of the proof:
Use factorization method by Da Prato, Zabczyk, Kwapien exploiting the identity

$$\int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha} ds = \frac{\pi}{\sin \pi\alpha}$$

for $\alpha \in (0,1)$, one has due to Fubini's theorem

$$\int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s$$
$$= \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} (-A)^{\delta/2} S_{t-s} \underbrace{\left(\int_0^s (s-\sigma)^{-\alpha} S_{s-\sigma} B(u_\sigma) dW_\sigma \right)}_{=: Y_s} ds$$

Moreover, use the inequality

$$||(-A)^{\delta/2} S_{t-s}|| \leq C \frac{1}{(t-s)^{\delta/2}}$$

The limit case $\delta = 1$: Heuristics

What about the limit case $\delta = 1$, i.e. problems of the form

$$\begin{cases} du & = \Delta u dt + (-\Delta)^{1/2} \sum_{k=1}^{d} B_k(u_t) dW_t^k \\ u(0) & = u_0 \in L^p(\mathbb{T}^N) \end{cases}$$

Heuristically, looking at the problem in Fourier basis, the Laplacian makes us gain k^2, the noise potentially lose k, yet due to Itô Isometry, one has to square, so we lose potentially in order k^2. In order for the equation to still admit a solution, one needs to thus require that the non-linearities B_k are "small" in the following sense.

The limit case $\delta = 1$: Uniform bounds

In particular, $(u^\delta)_\delta \subset L^2(\Omega \times [0,T], \mathbb{T}^N)$, i.e. using Itô's formula

$$||u_t||^2_{L^2(\mathbb{T}^N)} = ||u_0||^2_{L^2(\mathbb{T}^N)} - 2\int_0^t \int_{\mathbb{T}^N} |\nabla u|^2 dxds + \text{Martingale}$$
$$+ \int_0^t \sum_{i=1}^d \int_{\mathbb{T}^N} |(-\Delta)^{\delta/2} B_i(u_s)|^2 dxds$$

Lemma

Let $(B_k)_{k\le d}$ be as before and satisfy additionally

$$\left(\sum_{k=1}^d ||(-\Delta)^{\delta/2} B_k(u)||^2_{L^2(\mathbb{T}^N)}\right) \le C||\nabla u||^2_{L^2(\mathbb{T}^N)}$$

for $C < 2$. Then $(u^\delta)_\delta$ is uniformly bounded in $L^2(\Omega \times [0,T] \times \mathbb{T}^N)$

The limit case $\delta = 1$: Using Banach-Alaoglu

By Banach-Alaoglu, we conclude the existence of a weakly convergent subsequence in the reflexive Banach space $L^2(\Omega \times [0, T] \times \mathbb{T}^N)$. In what sense is now the original equation solved?

The limit case $\delta = 1$: Weak-mild solutions

Definition (Weak-mild solutions)

We call $u \in L^2(\Omega \times [0,T] \times \mathbb{T}^N)$ weak-mild solution to the problem

$$du = \Delta u dt + (-\Delta)^{1/2} B(u_t) dW_t \qquad u(0) = u_0 \tag{1}$$

if $\mathbb{P}$-almost surely one has for all $\xi \in D((-\Delta)^{1/2})$

$$\langle u_t, \xi \rangle = \langle S_t u_0, \xi \rangle + \int_0^t \langle (-\Delta)^{1/2} \xi, S_{t-s} B(u_s) dW_s \rangle$$

Lemma

The weak limit $u \in L^2(\Omega \times [0,T] \times \mathbb{T}^N)$ obtained above satisfies (1) in the weak-mild sense given that B is of linear growth.

The limit case $\delta = 1$: An application

In particular, one is able to apply the machinery presented to

$$\begin{cases} du & = \Delta u dt + \mathrm{div}\sigma(u_t) dW_t \\ u(0) & = u_0 \in L^p(\mathbb{T}^N) \end{cases}$$

in order to construct weak-mild solution via the approximate problems

$$\begin{cases} du & = \Delta u dt + (-\Delta)^{\delta/2} \underbrace{(-\Delta)^{-1/2} \mathrm{div}(\sigma(u))}_{=:B(u_t)} dW_t^k \\ u(0) & = u_0 \in L^p(\mathbb{T}^N) \end{cases}$$