Random walk on the torus

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Let ρ be a borelian probability measure on
$\mathbf{G} := \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\}$.
For any $x \in \mathbb{R}^2 \setminus \{0\}$ we consider the random walk defined by

$$\begin{cases} X_0 & = & x \\ X_{n+1} & = & g_{n+1} X_n \end{cases}$$

where (g_n) is an iid sequence of law ρ.

As the norm is submultiplicative, it's logarithm is subadditive and so we expect $\frac{1}{n} \ln \|X_n\|$ to converge to something.

If the support of ρ is a subset of

$$\left\{ E_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$$

then, as

$$E_{b_1} E_{b_2} = E_{b_1 + b_2}$$

what we have is a random walk on $\mathbb{R}$ rather than a random walk on $\mathrm{SL}_2(\mathbb{R})$.

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So, we have to make assumptions on $\mathrm{supp}\rho$.

Definition

We say that a closed subgroup $\mathbf{H} < \mathbf{G}$ is *strongly irreducible* if it doesn't fix any finite union of lines in $\mathbb{R}^2$.

Example

$$\left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}\right\},$$

are not strongly irreducible (but the last one is irreducible)

We also don't want the group to be compact so we make the following

Definition

We say that a closed subgroup $\mathbf{H} < \mathbf{G}$ is *proximal* if there is some $h \in \mathbf{H}$ whose spectral radius $r(h)$ satisfies $r(h) > 1$.

Remark

This is stronger than asking the group generated by $\mathrm{supp}\rho$ to be non compact (consider a unipotent matrix)

Example

If $\rho = \frac{1}{2}\delta_A + \frac{1}{2}\delta_B$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

then $\mathrm{supp}\rho$ generates a strongly irreducible and proximal subgroup : it is proximal since both matrices are proximal and strongly irreducible since the matrices are diagonalisable and have different eigenvectors.

In the sequel, we will note $\mathbf{G}_\rho$ the closure of the subgroup of $\mathbf{G}$ generated by the support of ρ (it is still a group and it is strongly irreducible and proximal if and only if so is the group generated by the support of ρ)

Theorem (Furstenberg-Kesten, ...)

Let ρ be a borelian probability measure on $\mathbf{G}$ *such that* $\int_{\mathbf{G}} |\ln \|g\|| \mathrm{d}\rho(g)$ *is finite and* $\mathbf{G}_\rho$ *is strongly irreducible and proximal.*
Then, there is $\lambda_1 > 0$ *such that*

$$\lim_n \frac{1}{n} \ln \|g_n \dots g_1\| = \lambda_1 \quad \rho^{\otimes \mathbb{N}}\text{-ae.}$$

Moreover, for any $x \in \mathbb{R}^2 \setminus \{0\}$,

$$\lim_n \frac{1}{n} \ln \|g_n \dots g_1 x\| = \lambda_1 \quad \rho^{\otimes \mathbb{N}}\text{-ae.}$$

In particular, the walk on $\mathbb{R}^2 \setminus \{0\}$ is transient.

Remark

If we add moment conditions on ρ, we can get the central limit theorem, the law of the iterated logarithm and large deviations inequalities.

From now on, we note $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This groups acts on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and this allows us to define a random walk : we fix a probability measure ρ on $\mathrm{SL}_2(\mathbb{Z})$, and then, starting at some point $x \in \mathbb{T}^2$, we consider

$$\left\{ \begin{array}{rcl} X_0 & = & x \\ X_{n+1} & = & g_{n+1} X_n \end{array} \right.$$

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Rational points have a particular behaviour : 0 is fixed. More generally, if we start at $\frac{p}{q} \in \mathbb{Q}^d/\mathbb{Z}^d$, then the walk stays in $\frac{1}{q}\mathbb{Z}^d/\mathbb{Z}^d$. So we are just looking at the random walk on a finite set (and in particular, (X_n) equidistributes in the orbit of x)

What happens if the starting point is irrational ?

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Theorem (Bourgain, Furmann, Lindenstrauss and Mozes)

Let ρ be a probability measure on Γ whose support generates a strongly irreducible and proximal subgroup and such that for some $\varepsilon \in \mathbb{R}_+^*$, $\sum_{\gamma\in\Gamma} \|\gamma\|^\varepsilon \rho(\gamma)$ is finite.
Note $\nu =$ Lebesgue's measure on $\mathbb{T}^2$.
Then, for any non rational point $x \in \mathbb{T}^2$ and any continuous function f,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to \int f \,\mathrm{d}\nu \;\; \rho^{\otimes\mathbb{N}}\text{-a.e.}$$

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Theorem

Same assumptions. For any $\gamma \in]0,1]$ *there is* $\beta_0 \in \mathbb{R}_+^*$ *such that for any* $B \in \mathbb{R}_+^*$ *and any* $\beta \in]0, \beta_0[$ *we have that for any* $x \in \mathbb{T}^2$ *such that*

$$d\left(x, \frac{p}{q}\right) \leqslant e^{-Bq^\beta}$$

has only finitely many solutions $\frac{p}{q} \in \mathbb{Q}^d/\mathbb{Z}^d$ *we have that for any* $\gamma-$*hölder-continuous function* f *on the torus there is* $\sigma^2(f) \in \mathbb{R}_+$ *such that*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(X_k) - \int f \mathrm{d}\nu \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2(f))$$

where we noted $\mathcal{N}(0,0)$ *the Dirac mass at* 0*.*

In particular (with the Jarník-Besicovitch theorem), the Hausdorff dimension of the set of points where the theorem doesn't hold is 0.

How to prove such a theorem ?

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We note P the Markov operator associated to the walk :

$$Pf(x) = \sum_{\gamma \in \Gamma} f(\gamma x)\rho(\gamma)$$

If $f = g - Pg$ with $g \in \mathcal{C}^0(\mathbb{T}^2)$, then,

$$\sum_{k=0}^{n-1} f(X_k) = \sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k) + g(X_0) - g(X_n)$$

and, $M_n = \sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k)$ is a martingale with bounded increments so we can use the classical CLT.

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$\Longrightarrow$ the idea (which is called Gordin's method) is to solve Poisson's equation $f = g - Pg + \int f \mathrm{d}\nu$.

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We can prove that $g \in \mathrm{L}^2(\mathbb{T}^2, \nu)$ and that, actually, if $f = g - Pg$, then the variance is given by

$$\sigma^2(f) = \int_{\mathbb{T}^2} |g|^2 - |Pg|^2 \mathrm{d}\nu$$

Moreover, if g is continuous and $\sigma^2(f) = 0$, then, for any $x \in \mathbb{T}^2$ and $\rho-$a.e. $(g_n) \in \Gamma^{\mathbb{N}}$,

$$\sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k) = 0$$

In particular,

$$\sum_{k=0}^{n-1} f(X_k) = g(X_0) - g(X_n)$$

so, we get that if $f = g - Pg$ with g continuous is such that $\sigma^2(f) = 0$, then, for any $x \in \mathbb{T}^2$,

$$\sum_{k=0}^{n-1} f(X_k)$$

is bounded in $\mathrm{L}^\infty(\mathbb{P}_x)$.

The same works on $\mathbb{T}^d$ with the same strong-irreducibility and proximality assumptions.

If we add translations by diophantine numbers the situation is much more simple since then we don't have the problem of finite orbits and we can prove that $P^n f(x)$ converges exponentially fast to $\int f \mathrm{d}\nu$ for any Hölder continuous function. In particular, we also have the CLT, LIL and even a LDP.

Merci de votre attention !