Liouville Quantum Gravity on Riemann Surfaces

Guillaume Remy

École Normale Supérieure

May 17, 2016

Field Theory: a general framework

Quantum Field Theory: a general framework in physics

Goal: compute correlations of fields: $\langle \prod_{i \in I} \phi_i(z_i) \rangle$

Conformal Field Theory: conformal invariance in 2D

Continuum limit of the Ising model

Fields: spin operator

Liouville Quantum Gravity:

Polyakov, "Quantum Geometry of Bosonic Strings"

Brownian motion seen as a path integral

Space of paths: $\Sigma = \{\sigma : [0,1] \to \mathbb{R}, \sigma(0) = 0\}$

Action functional: $S_{BM}(\sigma) = \frac{1}{2}\int_0^1 |\sigma'(r)|^2 dr$

$$\mathbb{E}[F((B_s)_{0 \leq s \leq 1})] = \frac{1}{Z}\int_\Sigma D\sigma F(\sigma) e^{-S_{BM}(\sigma)}$$

$D\sigma$: formal uniform measure on Σ

Classical Theory / Quantum Theory

Minimum of S_{BM} $\rightarrow$ straight line $=$ classical solution
Path integral $\rightarrow$ Brownian motion $=$ quantum correction

Some definitions

Let (M, g) be a Riemann surface with a metric g.

Metric tensor g : $M \to S_2^+(\mathbb{R})$

- Length of a curve $z = z^i(t)$: $\int_a^b \sqrt{g_{ij}(z(t))\frac{dz^i}{dt}\frac{dz^j}{dt}}dt$
- Area of A: $\int_A \sqrt{\det g(x)}dx^2 = \int_A \lambda_g(dx)$
- Scalar curvature R_g:

$$\text{For } g(x) = \begin{pmatrix} e^{f(x)} & 0 \\ 0 & e^{f(x)} \end{pmatrix}, \; R_g = -e^{-f}\Delta f.$$

Spherical metric on $\mathbb{R}^2$: $g(x) = \frac{4}{(1+|x|^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R_g = 2$.

Classical Liouville Theory

For all maps $X : M \to \mathbb{R}$, we define:

$$S_L(X,g) = \frac{1}{4\pi}\int_M (|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X})\lambda_g$$

- $|\partial^g X|^2 = \sum_{i,j} g_{ij}\partial^i X \partial^j X$
- $Q, \gamma, \mu > 0$ positive constants

Uniformization of (M, g)

Assume X_{min} to be the minimum of S_L and define $g' = e^{\gamma X_{min}} g$. Then $R_{g'} = -2\pi\mu\gamma^2$ if we choose $Q = \frac{2}{\gamma}$.
$\Longrightarrow$ *The minimum of S_L provides a metric of constant negative curvature.*

Defining Liouville Quantum Gravity

Formal definition

Random metric $e^{\gamma\phi}g$ where the law of ϕ is given by:
$\mathbb{E}[F(\phi)] = \frac{1}{Z}\int F(X)e^{-S_L(X,g)}DX$

First goal: give a meaning to ϕ for different M.

- $M =$ Riemann sphere: David-Kupiainen-Rhodes-Vargas
- $M =$ Torus or higher genus: David-Guillarmou-Rhodes-Vargas
- $M =$ Unit disk: Huang-Rhodes-Vargas
- $M =$ Annulus: Remy

$\phi =$ Liouville field

Why the Liouville action?

- $|\partial^g X|^2$: analogue of the $|\sigma'|^2$ for Brownian motion

  $\frac{1}{Z}\int F(X)e^{-\frac{1}{4\pi}\int_M |\partial^g X|^2 \lambda_g} DX$: formally defines the law of the Gaussian Free Field (GFF)

  GFF: Gaussian process with covariance function given by the Green function of the Laplacian $-\Delta_g$

- $QR_g X$: curvature term

- $\int_M e^{\gamma X}\lambda_g$ = area of M in the metric $g' = e^{\gamma X} g$
  $\Rightarrow$ penalizes large areas
  $\Rightarrow$ required to have a well defined Liouville field

Liouville measure

- ϕ is a random distribution $\Rightarrow$ difficult to define $e^{\gamma\phi}$
- Well defined Liouville measure $Z(A) = \int_A e^{\gamma\phi}\lambda_g$

  Conjectured limit of uniform planar maps for $\gamma = \sqrt{\frac{8}{3}}$

  Conjectured limit of planar maps with an Ising model for $\gamma = \sqrt{3}$

Discrete model / Continuum limit

Brownian motion = scaling limit of random walks
Liouville quantum gravity = limit of discrete 2D models (like planar maps)

Insertion points

- For $M = \mathbb{S}^2$, Gauss-Bonnet: $\int_{\mathbb{S}^2} R_g \lambda_g = 8\pi > 0$
- No metric of constant negative curvature
  $\Rightarrow S_L$ has no minimum
  $\Rightarrow \frac{1}{Z} \int F(X) e^{-S_L(X,g)} DX$ not defined
- Instead we consider:
  $\frac{1}{Z} \int F(X) e^{\sum_{i=1}^{n} \alpha_i X(z_i)} e^{-S_L(X,g)} DX = \langle \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \rangle$
  $=$ correlation function of the fields $e^{\alpha_i X(z_i)}$
- (z_i, α_i): insertion points $=$ singularities of the metric
- For $\mathbb{S}^2$: at least 3 insertions required