On Optimal Skorokhod Embedding

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Outline

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1. Introduction

2. Optimal Skorokhod Embedding Problem

Skorokhod embedding problem

- Given
  - $B = (B_t)_{t \geq 0}$ be a Brownian motion (BM) defined on $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$;
  - μ a centered probability distribution on $\mathbb{R}$.
- The Skorokhod embedding problem (SEP) aims to find an $\mathbb{F}-$stopping time τ s.t.
  - $B_{\tau \wedge \cdot} := (B_{\tau \wedge t})_{t \geq 0}$ is uniformly integrable (UI) ;
  - $B_\tau \sim \mu$.
- Two existing formulations exist in the literature :
  - "Strong" embedding : $\mathbb{F} = \mathbb{F}^B$;
  - "Weak" embedding : $\mathbb{F} \supset \mathbb{F}^B$.

"Optimal" embeddings

• A fruitful idea : compare the realization of a Brownian trajectory with the realization of some well-controlled process $(\phi_t(B))_{t\geq 0}$ and use the latter to decide when to stop the former :

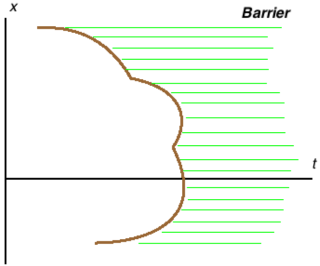
$$(B_t,\ t),\ \ (B_t,\ \sup_{s\leq t} B_s),\ \ (B_t,\ L_t),\ \cdots$$

• Famous embeddings : Skorokhod, Root, Rost, Azéma-Yor, Jacka, Monroe, Vallois, Cox-Hobson, etc.

• A number of the above embeddings satisfy some particular "optimality".

A motivating example : Root's embedding (I)

- A Borel set $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}$ is called a barrier if $(s, x) \in \mathcal{R}$ and $s < t$ implies $(t, x) \in \mathcal{R}$.



- There exists a barrier $\mathcal{R}$ s.t. the SEP is solved by the stopping time

$$\tau_{Root} \;:=\; \inf \left\{ t \in \mathbb{R}_+ : \; (t, B_t) \in \mathcal{R} \right\}.$$

A motivating example : Root's embedding (II)

• Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be a strictly concave function, then the stopping time τ_{Root} solves the following optimization problem :

$$\sup_{\tau:\ \mu\text{-embedding}} \mathbb{E}[\Phi(\tau)] = \mathbb{E}[\Phi(\tau_{Root})].$$

Remark

- *The strong and weak formulations are equivalent to study this optimization problem.*
- *The optimality of Root's embedding typically used the particular structure of* Φ.
- *What happened for a general* $\Phi = \Phi\big((B_t)_{t\le\tau}, \tau\big)$ *?*

Outline

Probabilistic formulation of SEP (I)

- Let Ω be the space of continuous functions $\omega = (\omega_t)_{t \geq 0}$ s.t. $\omega_0 = 0$.
- Define the enlarged space $\overline{\Omega} := \Omega \times \mathbb{R}_+$ and denote by $\bar{\omega} = (\omega, \theta)$ its elements.
- Define the canonical element (B, T) by $B(\bar{\omega}) = \omega$ and $T(\bar{\omega}) = \theta$.
- Denote by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \geq 0}$ the canonical filtration given by

$$\overline{\mathcal{F}}_t \ := \ \sigma(B_s, s \leq t) \vee \sigma\left(\{T \leq s\} \text{ for all } s \in [0, t]\right).$$

- In particular, T is an $\overline{\mathbb{F}}-$stopping time.

Probabilistic formulation of SEP (II)

• Let $\overline{\mathcal{P}}$ be the set of probability measures $\overline{\mathbb{P}}$ on $\overline{\Omega}$ s.t. B is an $\overline{\mathbb{F}}-$BM under $\overline{\mathbb{P}}$ and $B_{T\wedge\cdot}$ is UI.

• Let μ be a zero-mean probability distribution on $\mathbb{R}$, i.e. $\mu(|x|) < +\infty$ and $\mu(x) = 0$. Here we denote for all measurable functions $\lambda : \mathbb{R} \to \mathbb{R}$

$$\mu(\lambda) \quad := \quad \int \lambda d\mu.$$

• Denote by $\overline{\mathcal{P}}(\mu) \subset \overline{\mathcal{P}}$ be the subset of measures $\overline{\mathbb{P}}$ s.t. $B_T \overset{\overline{\mathbb{P}}}{\sim} \mu$.

Optimal SEP : Primal and dual problems (I)

• Let $\Phi : \overline{\Omega} \to \mathbb{R}$ be a measurable function. Φ is called non-anticipative if

$$\Phi(\omega, \theta) \quad = \quad \Phi(\omega_{\theta \wedge \cdot}, \theta) \text{ for all } (\omega, \theta) \in \overline{\Omega}.$$

• For a non-anticipative function Φ, the optimal SEP is defined by

$$P(\mu) \quad := \quad \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu)} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi(B, T)\big].$$

• Let Λ be the space of continuous functions $\lambda : \mathbb{R} \to \mathbb{R}$ with linear growth.

• Let $\mathbb{F}^B = (\mathcal{F}_t^B)_{t \geq 0}$ be the natural filtration of B and $\mathbb{P}_0$ be the Wiener measure on Ω.

Optimal SEP : Primal and dual problems (II)

• $\mathcal{H}$ the collection of all $\mathbb{F}-$predictable processes $H : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ s.t.

- $(H \cdot B) := \int_0^\cdot H_t dB_t$ is a $\mathbb{P}_0-$martingale ;
- $(H \cdot B)_t \geq -C(1 + |B_t|)$ for some constant $C > 0$.

• Denote

$$\mathcal{D} := \Big\{(\lambda, H) \in \Lambda \times \mathcal{H} : \lambda(\omega_t) + (H \cdot B)_t \geq \Phi(\omega, t), \text{ for all } t \in \mathbb{R}_+ \text{ and } \mathbb{P}_0 \text{ - a.e.} \omega \in \Omega\Big\}.$$

• The dual problem is given by

$$D(\mu) := \inf_{(\lambda, H) \in \mathcal{D}} \mu(\lambda).$$

Duality result

Theorem (GG & Tan & Touzi)

Assume that the non-anticipative function $\Phi : \overline{\Omega} \to \mathbb{R}$ is bounded from above, and $\theta \mapsto \Phi(\omega_{\theta_n \wedge \cdot}, \theta)$ is upper-semicontinuous for $\mathbb{P}_0$ -a.e. $\omega \in \Omega$. Then there exists $\overline{\mathbb{P}}^* \in \overline{\mathcal{P}}(\mu)$ s.t.

$$\mathbb{E}^{\overline{\mathbb{P}}^*}[\Phi(B, T)] \; = \; P(\mu) \; = \; D(\mu).$$

Remark

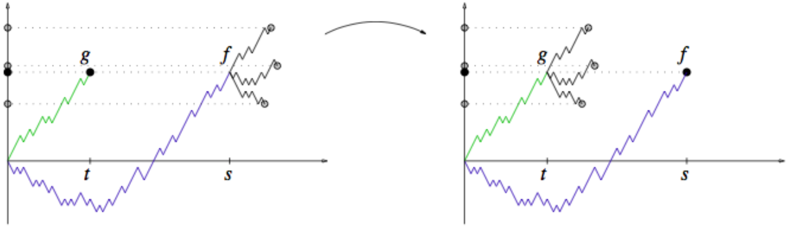
- *In view of Dubins-Dambis-Schwarz's Theorem, the theorem above yields the Kantorovich duality for continuous martingale optimal transport problem.*
- *The duality allows to derive a geometric characterization of optimizers.*

Stop-Go pair

- Let $\overline{\Omega}_+ := \{\bar{\omega} = (\omega, \theta) \in \overline{\Omega} : \theta > 0\}$.
- $(\bar{\omega}, \bar{\omega}') \in \overline{\Omega} \times \overline{\Omega}$ is called a Stop-Go pair if $\omega_\theta = \omega'_{\theta'}$ and

$$\Phi(\bar{\omega}) + \Phi(\bar{\omega}' \otimes \bar{\omega}'') > \Phi(\bar{\omega} \otimes \bar{\omega}'') + \Phi(\bar{\omega}') \text{ for all } \bar{\omega}'' \in \overline{\Omega}_+.$$

- Denote by SG the set of Stop-Go pairs.



Monotonicity principle

- Let $\Gamma \subset \overline{\Omega}$, define

$$\Gamma^< \;:=\; \left\{(\omega,\theta) \in \overline{\Omega} :\ \exists\, (\omega',\theta') \in \Gamma \text{ s.t. } \theta < \theta' \text{ and } \omega_{\theta\wedge\cdot} = \omega'_{\theta\wedge\cdot}\right\}.$$

Theorem (Beiglböck & Cox & Huesmann, G. & Tan & Touzi)

Assume that the duality holds and let $\overline{\mathbb{P}}^*$ be an optimizer, then there is a Borel set $\Gamma \subset \overline{\Omega}$ s.t. $\overline{\mathbb{P}}^*(\Gamma) = 1$ and $\mathrm{SG} \cap \left(\Gamma^< \times \Gamma\right) = \emptyset$.

Remark

- *Consider two paths (ω,θ) and (ω',θ') which end at the same level, i.e. $\omega_\theta = \omega'_{\theta'}$. We want to determine which of the two paths should be "stopped" and which one should be allowed to "go" on further.*
- *The condition $\omega_\theta = \omega'_{\theta'}$ is necessary to guarantee that a modified stopping rule still embeds the measure μ.*

Back to Root's embedding

Theorem

Let $\overline{\mathbb{P}}^*$ be an optimizer, then there exists a barrier $\mathcal{R}$ s.t.

$$T \;:=\; \inf\left\{t \geq 0 : (t, B_t) \in \mathcal{R}\right\},\; \overline{\mathbb{P}}^* \text{ - a.s.}$$

Proof. Pick, by monotonicty principle, a set $\Gamma \subset \overline{\Omega}$ s.t. $\overline{\mathbb{P}}^*$ - almost surely, $(B, T) \in \Gamma$. By concavity of Φ, the set of Stop-Go pairs is given by

$$\mathrm{SG} \;=\; \left\{\big((\omega,\theta),(\omega',\theta')\big) :\; \omega_\theta = \omega'_{\theta'} \text{ and } \theta < \theta'\right\}.$$

As $\mathrm{SG} \cap (\Gamma^< \times \Gamma) = \emptyset$, define the barrier by

$$\mathcal{R} \;:=\; \{(t,x) :\; \exists\, (\omega,\theta) \in \Gamma \text{ s.t. } \omega_\theta = x \text{ and } t < \theta\},$$

then $\cdots$

More remarks

- There exists a Borel set SG^* depending on $\overline{\mathbb{P}}^*$ s.t.

$$\mathrm{SG} \cap \left(\Gamma^< \times \Gamma\right) \;\subseteq\; \mathrm{SG}^* \cap \left(\Gamma^< \times \Gamma\right) \;=\; \emptyset.$$

- We may extend the analysis to multiple marginal case, i.e.

$$\overline{\Omega} \;:=\; \Omega \times \mathbb{R}_+^m \quad \text{and} \quad (B, T_1, \cdots, T_m)$$

$$\overline{\mathcal{P}}(\mu_1, \cdots, \mu_m) \quad := \quad \left\{ \overline{\mathbb{P}} \in \overline{\mathcal{P}} : B_{T_k} \overset{\overline{\mathbb{P}}}{\sim} \mu_k \text{ for all } k = 1, \cdots, m \right\}.$$

Thank you for your attention !