Voronoi diagram on a Riemannian surface

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Motivation

Aim : Show a link between mean characteristics of the Voronoi cells and local characteristics of the surface

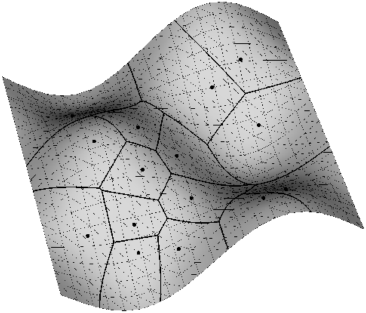


image:R.Kunze

Framework

- S Riemannian surface, with its Riemannian metric d,
- dx area measure induced by the metric,
- Φ Poisson point process of intensity λdx and $x_0 \in S$ added to Φ,
- The Voronoi cell of x_0 defined by

$$C(x_0, \Phi) = \{y \in S, d(x_0, y) \leq d(x, y), \forall x \in \Phi\}$$

- N the number of vertices.

Outline

1. Case of the sphere
2. Arbitrary surface

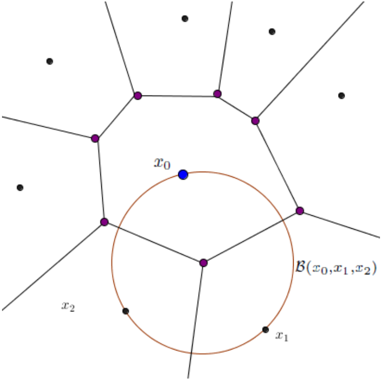
Mean number of vertices

wlog, assume x_0 to be the North pole on the sphere of constant curvature K (of radius $\frac{1}{\sqrt{K}}$)

$$\mathbb{E}[N(\mathcal{C})] = 6 - \frac{3K}{\pi\lambda} + e^{-\frac{4\pi\lambda}{K}}\left(\frac{3K}{\pi\lambda} + 6\right)$$

Miles (1971) : n uniform points on the sphere

Sketch of proof



Step 1: characterize vertices of $\mathcal{C}$

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \mathbb{E}\left[\sum_{x_1, x_2 \in \Phi} \mathbb{1}_{\{\mathcal{B}_1(x_0, x_1, x_2) \cap \Phi = \emptyset\}} + \mathbb{1}_{\{\mathcal{B}_2(x_0, x_1, x_2) \cap \Phi = \emptyset\}}\right]$$

Step 1: characterize vertices of $\mathcal{C}$

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \iint_{x_1, x_2 \in \mathcal{S}(K)} \left(e^{-\lambda \operatorname{vol}(\mathcal{B}_1(x_0, x_1, x_2))} + e^{-\lambda \operatorname{vol}(\mathcal{B}_2(x_0, x_1, x_2))} \right) dx_1 dx_2$$

Step 2: apply Mecke-Slivnyak formula

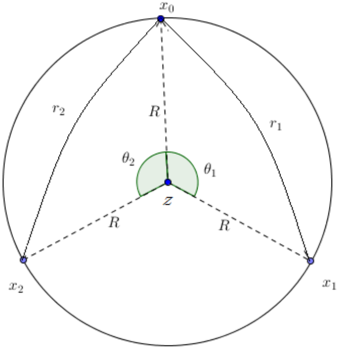
Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \int_{r_1,\varphi_1,r_2,\varphi_2} \left(e^{-\lambda \operatorname{vol}(\mathcal{B}_1(x_0,x_1,x_2))} + e^{-\lambda \operatorname{vol}(\mathcal{B}_2(x_0,x_1,x_2))} \right)$$

$$\times \frac{\sin(\sqrt{K} r_1)}{\sqrt{K}} \frac{\sin(\sqrt{K} r_2)}{\sqrt{K}} dr_1 d\varphi_1 dr_2 d\varphi_2$$

Step 3: use spherical coordinates

Sketch of proof



$$r_1 = \frac{2}{\sqrt{K}} \arcsin(\sin\left(\frac{\theta_1}{2}\right)\sin(\sqrt{K}R))$$

$$r_2 = \frac{2}{\sqrt{K}} \arcsin(\sin\left(\frac{\theta_2}{2}\right)\sin(\sqrt{K}R))$$

$$\varphi_1 = \varphi + \frac{\pi}{2} - \arctan(\tan\left(\frac{\theta_1}{2}\right)\cos(\sqrt{K}R))$$

$$\varphi_2 = \varphi + \frac{\pi}{2} - \arctan(\tan\left(\frac{\theta_2}{2}\right)\cos(\sqrt{K}R))$$

Step 4: make a Blaschke-Petkantschin type change of variables

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = 4\pi\lambda^2 I \int_0^{\frac{\pi}{2\sqrt{K}}} \left(e^{-\lambda \frac{2\pi}{K}(1-\cos(\sqrt{K}R))} + e^{-\lambda \frac{2\pi}{K}(1+\cos(\sqrt{K}R))} \right) \frac{\sin^3(\sqrt{K}R)}{\sqrt{K}} dR$$
$$= 6 - \frac{3K}{\pi\lambda} + e^{-\frac{4\lambda\pi}{K}} \left(6 + \frac{3K}{\lambda\pi} \right)$$

where

$$I = \int_{\theta_1,\theta_2 \in [0,2\pi]} \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \left| \sin\left(\frac{\theta_1-\theta_2}{2}\right) \right| d\theta_1 d\theta_2$$

Step 4: Make a Blaschke-Petkantschin type change of variables

Strategy

Find a way to adapt the method to a general surface

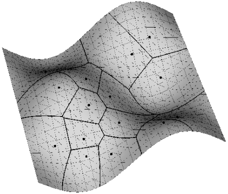


image:R.Kunze

- **Step 1:** characterize vertices of $\mathcal{C}$
- **Step 2:** apply Mecke-Slivnyak formula
- **Step 3:** use geodesic polar coordinates
- **Step 4:** make a Blaschke-Petkantschin type change of variables
- **Step 5:** find the volume of a geodesic ball

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \mathbb{E}\left[\sum_{x_1,x_2\in\Phi}\ \sum_{\text{circumscribed balls}} \mathbb{1}_{\{\mathcal{B}(x_0,x_1,x_2)\cap\Phi=\emptyset\}}\right]$$

Step 1: characterize vertices of $\mathcal{C}$

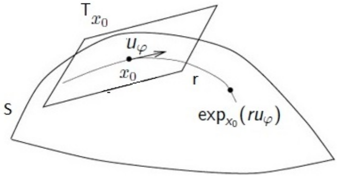
Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \iint_{x_1, x_2 \in S} \sum_{\text{circumscribed balls}} e^{-\lambda \,\mathrm{vol}(\mathcal{B}(x_0, x_1, x_2))} dx_1 \, dx_2$$

1. Points "far" from x_0 contribute negligibly.
2. For points around x_0, we need similar changes of variables.

Step 2: apply Mecke Slivnyak formula

Exponential map



Around x_0, S can always be parametrized by its geodesic polar coordinates (r, φ), ie

$$x = \exp_{x_0}(ru_\varphi)$$

Step 3: use geodesic polar coordinates

Rauch theorem

$$dx = f(r,\varphi)drd\varphi$$

Let K denote the Gaussian curvature.

Rauch theorem (1951)

Si $0 < \delta \leq K \leq \Delta$

$$\frac{\sin(\sqrt{\Delta}r)}{\sqrt{\Delta}} \leq f(r,\varphi) \leq \frac{\sin(\sqrt{\delta}r)}{\sqrt{\delta}}$$

Application: $\delta = K(x_0) - \varepsilon$, $\Delta = K(x_0) + \varepsilon$

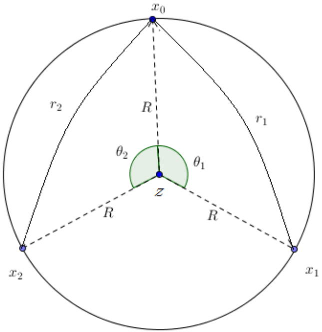
Step 3: use geodesic polar coordinates

Sketch of proof

$$E[N(\mathcal{C})] = \frac{\lambda^2}{2} \int_{\substack{(r_1,\varphi_1)\\(r_2,\varphi_2)}} e^{-\lambda \, \mathrm{vol}(\mathcal{B}(x_0,x_1,x_2))} \times \left(r_1 - \tfrac{K(x_0) r_1^3}{6} + o(r_1^3)\right)\left(r_2 - \tfrac{K(x_0) r_2^3}{6} + o(r_2^3)\right) dr_1 d\varphi_1 dr_2 d\varphi_2 + O\left(e^{-c\lambda}\right)$$

Step 3: use geodesic polar coordinates

Sketch of proof



$r_1 =?$

$r_2 =?$

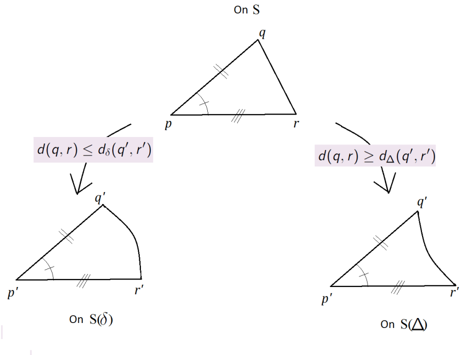
$\varphi_1 =?$

$\varphi_2 =?$

Step 4: make a Blaschke-Petkantschin type change of variables

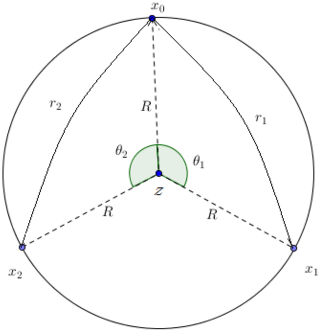
Toponogov theorem

If $\delta \leq K \leq \Delta$



Step 4: make a Blaschke-Petkantschin type change of variables

Sketch of proof



$$r_1 = 2\sin(\theta_1/2)R - \frac{K(x_0)R^3}{3}\sin(\theta_1/2)\cos^2(\theta_1/2) + o(R^3)$$

$$r_2 = 2\sin(\theta_2/2)R - \frac{K(x_0)R^3}{3}\sin(\theta_2/2)\cos^2(\theta_2/2) + o(R^3)$$

$$\varphi_1 = \varphi + \frac{\pi}{2} - \frac{\theta_1}{2} + \frac{K(x_0)R^2}{4}\sin(\theta_1) + o(R^2)$$

$$\varphi_2 = \varphi + \frac{\pi}{2} - \frac{\theta_2}{2} + \frac{K(x_0)R^2}{4}\sin(\theta_2) + o(R^2)$$

Step 4: make a Blaschke-Petkantschin type change of variables

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = 2\lambda^2 I \int_{\varphi} \int_{R} e^{-\lambda \operatorname{vol}(\mathcal{B}(z,R))} \left(R^3 - \tfrac{K(x_0)R^5}{2} + o(R^5) \right) dR d\varphi + O(e^{-c\lambda})$$

where

$$I = \int_{\theta_1,\theta_2} \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \left| \sin\left(\frac{\theta_1-\theta_2}{2}\right) \right| d\theta_1 d\theta_2$$

Step 4: make a Blaschke-Petkantschin type change of variables

Volume of small geodesic balls

Bertrand-Diquet-Puiseux theorem (1848)

When $r \to 0$, $x \in S$

$$\mathrm{vol}(\mathcal{B}(z,r)) = \pi r^2 - \frac{K(z)\pi}{12} r^4 + o(r^4)$$

Step 5: find the volume of the circumscribed ball

Result

$$\mathbb{E}[N(\mathcal{C})] = 12\pi^2\lambda^2 \int_0^{R_{max}} e^{-\lambda(\pi R^2 - \frac{\pi K(x_0) R^4}{12} + o(R^4))} \times [R^3 - \frac{K(x_0)R^5}{2} + o(R^5)]dR + O(e^{-c\lambda})$$

When λ goes to infinity, Laplace's method yields

Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = 6 - \frac{3K(x_0)}{\pi\lambda} + o\left(\frac{1}{\lambda}\right)$$

Take Home Message

- **On surfaces:**
  - $\hookrightarrow$ Link between mean number of vertices and Gaussian curvature
  - $\hookrightarrow$ Result available for surface of negative curvature (Isokawa 2000)
  - $\hookrightarrow$ Other mean characteristics: area, perimeter
- **Ongoing work on dimension ≥ 3:**
  - $\hookrightarrow$ Link between mean number of vertices and scalar curvature
  - $\hookrightarrow$ Perspective: other characteristics to get other curvatures
  - $\hookrightarrow$ . . .

Thank you for your attention!